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Relativistic quantum kinematics in the Moyal representation

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Abstract. In this paper, we obtain the phase-space quantisation for relativistic spinning particles. The main tool is what we call a 'Stratonovich-Weyl quantiser' which relates functions on phase space to operators on a suitable Hilbert space, and has the essential properties of covariance (under a group representation) and traciality. Our phase spaces are coadjoint orbits of the restricted Poincaré group; we compute and explicitly coordinatise the orbits corresponding to massive particles, with or without spin. Some orbits correspond to unitary irreducible representations of the Poincaré group; we show that there is a unique Stratonovich-Weyl quantiser from each of these phase spaces to operators on the corresponding representation spaces, and compute it explicitly.

We develop the formalism by computing relativistic Wigner functions and twisted products for Klein-Gordon particles; these Wigner functions are supported on the mass shell. We thereby obtain an expression for the position probability density which is local, i.e. free from the difficulty of supraluminal propagation of the usual position probability density. It is shown explicitly how observables on phase space may be quantised; for example, we prove that the canonical position coordinate corresponds to the Newton-Wigner position operator, irrespective of spin.

We show how relativistic phase-space quantisation applies to particles governed by the Dirac equation. In effect, we construct a Stratonovich-Weyl quantiser whose associated Hilbert space is the space of positive-energy solutions of the Dirac equation.

1. Introduction

The work of Moyal [1] did much to clarify that the 'Weyl correspondence' [2] and the 'Wigner distribution' [3] are elements of the fourth formulation—historically speaking—of non-relativistic quantum mechanics. The Moyal formulation has not had, by far, a success comparable to those of Heisenberg, Schrödinger or Feynman. Let us note, however, that the difficulty of extending it to cover spinning or relativistic particles was one of the reasons for that relative lack of success.

We consider in this paper *elementary* systems. The phrase 'elementary quantum system' is usually taken to mean an irreducible (projective) representation of some invariance group of physical interest, such as the Galilei or Poincaré groups; the rays of the Hilbert space of the representation are taken as the states of the quantum system, and its observables are operators on this Hilbert space. The concept became firmly established following the landmark paper by Wigner [4] about the representations of the Poincaré group.

In the Moyal formulation of quantum mechanics, a different point of view is adopted: both states and observables are real functions (or generalised functions) on the classical phase space, and expected values are computed, as in classical statistical physics, by averaging over the phase space.

To be precise, the rule for computing the expected values of an observable f in a state ρ remains the classical rule:

$$\langle f\rangle_\rho = \frac{\int_M f(u)\rho(u)\,\mathrm{d}u}{\int_M \rho(u)\,\mathrm{d}u} \tag{1}$$

where, in the ordinary case, $M=\mathbb{R}^{2n}$ and $\mathrm{d}u$ is Lebesgue measure on $\mathbb{R}^{2n}$, n being the number of degrees of freedom of the system.

Observables are composed via the non-commutative *twisted product*, since the classical pointwise product of observables is excluded by the uncertainty principle, which forbids localisation at a point of phase space. States are defined as the *positive functionals* of the twisted product algebra, i.e. ρ represents a state if

$$\int_M \bar{f}\times f(u)\rho(u)\,\mathrm{d}u \geqslant 0$$

for any f. This is parallel to what is done in classical statistical mechanics, with the twisted product substituting for the ordinary product, but here the states no longer need to correspond to non-negative functions (since the twisted product of a function with its complex conjugate can take negative values). It turns out that Wigner distributions are essentially the pure states of the twisted product algebra; and so, an intrinsically *autonomous* theory can be established, equivalent to, but independent of, conventional quantum mechanics.

In this paper, we construct the Moyal representation of relativistic quantum theory. This arises out of a programme for the Moyal quantisation of general phase spaces. Although we want to consider here only the physically relevant problem of relativistic mechanics, some points of principle concerning this programme must be made. Its backbone is the same as that of the 'geometric quantisation' programme, namely the coadjoint orbit picture introduced by Kirillov, Kostant and Souriau [5].

In modern renditions of classical mechanics, one considers a symplectic manifold M (or, more generally, a manifold with a Poisson bracket structure); the invariance group is a Lie group G acting on M by transformations which preserve this structure. We say we have an *elementary classical system* [6] if this action is transitive, i.e. if M is a homogeneous symplectic manifold (HSM) for the group G.

The elementary systems whose invariance group is the given (connected) Lie group G appear in the conventional approach to quantum mechanics as projective unitary representations of G. It is convenient to find and to present the projective unitary representations of G as linear unitary representations of another group $\bar{\mathrm{G}}$ which is a 'splitting group' for G [7, 8], which we shall describe in more detail below. The connection with the classical framework arises from the work of Kirillov [5], according to which there should be a correspondence between the unitary irreducible representations of $\bar{\mathrm{G}}$ and the orbits of the coadjoint action of $\bar{\mathrm{G}}$ on the dual space of its Lie algebra. Experience suggests that not all coadjoint orbits are eligible, but only those which satisfy certain integrality conditions. (A simple formulation of integrality conditions is found in [9].)

The phase-space quantisation programme may be formulated as follows. Let G be the physical invariance group whose elementary systems we want to study. Construct the appropriate splitting group $\bar{\mathrm{G}}$ and assume that Kirillov's paradigm works for $\bar{\mathrm{G}}$ (this is the case for most invariance groups of physical interest). The orbits of the coadjoint action of $\bar{\mathrm{G}}$ that are also HSM for G are the 'phase spaces'.

At this point we part company with the outlook and techniques of geometric quantisation and we seek to extend Kirillov's theory in a new direction. A *Moyal quantum elementary system* is a classical elementary system M plus a G-equivariant *twisted product* on spaces of functions on M, such that through (1) the physical expectations of the theory coincide with the ones on the Hilbert space of the representation associated with M. In practice, one frequently knows beforehand the representation theory of G, so that the simplest approach seems to be to link the coadjoint orbits of G directly with the operatorial theory, by means of an appropriate correspondence rule which yields the twisted product as the image of the usual composition of operators.

The programme just outlined was carried out by two of us for the group SU(2), yielding the Moyal representation of spin, entirely equivalent to the conventional one, but with some interpretative and computational advantages [10]. The ordinary Moyal theory was also reinterpreted as a theory of Galilean elementary systems in our sense and extended to particles with arbitrary spin [11].

Section 2 serves as a guideline for the whole paper. We first deal with geometric preliminaries pertaining to Kirillov theory; this part, together with section 3, may be taken as a primer on coadjoint orbit techniques; we think these belong in the toolkit of every theoretical physicist. Thereafter we introduce the key concept of Stratonovich–Weyl quantiser, which gives the aforementioned link between the phase spaces and the operatorial theory on Hilbert spaces. Finally, we go through two important examples of its use. The results of the first example—concerning pure spin systems—are employed throughout the paper. In section 3 the coadjoint action for the Poincaré group is computed. Section 4 is the heart of the paper: here we derive the (unique) Stratonovich–Weyl quantiser for relativistic spinning particles in the Wigner realisation.

Section 5 deals with the resulting Moyal formulation in the spinless (Klein–Gordon) case. In section 6, the operators corresponding to several important phase-space observables are computed in the Wigner realisation, for any spin. Section 7 constructs the Stratonovich–Weyl quantiser for particles described by the Dirac equation.

In the concluding section 8, we briefly review some recent attempts to derive 'relativistic Wigner functions' or 'relativistic Weyl transforms' and compare the results with those of our group-theoretic approach.

Throughout the paper, units of measure are taken so that $\hbar = 1$ and $c = 1$.

2. Phase-space quantisation in general

2.1. Geometric preliminaries

Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra. If one wishes to determine the projective unitary representations of G, the classical method proposed by Bargmann [12] may be used. It employs a family of extensions of the covering group of G, and in general different extensions must be used for different projective unitary representations. Some years ago, one of us [7] showed how this method could be improved, in the following sense: given G, a uniquely determined 'splitting group' $\bar{G}$ can be found such that *all* projective unitary representations of G can be lifted to unitary representations of $\bar{G}$. Actually, there exist in general several groups G′ and morphisms $\mu : G' \to G$ such that every projective representation of G can be lifted to a unitary representation of G′ mapping the elements of $\ker \mu$ into multiples of the identity. The particular construction proposed in [7] is very easy to handle but may not be 'minimal': see [8].

We recall that construction. Let $H^2(\mathfrak{g}, \mathbb{R}) \equiv \mathbb{R}^m$ be the second cohomology group of $\mathfrak{g}$ for the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$. Then we consider the central extension $\bar{\mathfrak{g}} = \mathbb{R}^m \oplus \mathfrak{g}$, with appropriate commutation relations; the connected and simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$ is the splitting group. For details, see [7, 8], where the somewhat inadequate name 'projective covering group' was used to refer to this 'splitting group'. Therein it is proved that there is a homomorphism $\mu : \bar{\mathrm{G}} \to \mathrm{G}$ such that each unitary representation of $\bar{\mathrm{G}}$ mapping the kernel of μ into the circle group U(1) induces a projective unitary representation of G; and conversely, each projective unitary representation of G can be lifted to a unitary representation of $\bar{\mathrm{G}}$. If $H^2(\mathfrak{g}, \mathbb{R}) = 0$, as happens for all semisimple groups, then $\bar{\mathrm{G}}$ reduces to the (universal) covering group $\tilde{\mathrm{G}}$ of G. Well known examples of covering groups in physics are SU(2), which covers the rotation group, and SL(2, $\mathbb{C}$), which covers the group $\mathscr{L}_+^\uparrow$ of proper orthochronous Lorentz transformations.

We recall that the adjoint representation of G is the map $\mathrm{Ad}: \mathrm{G} \to \mathrm{GL}(\mathfrak{g})$ defined by

$$\exp[t(\mathrm{Ad}\, g)X] = g(\exp tX)g^{-1}$$

for $g \in \mathrm{G}$, $X \in \mathfrak{g}$, $t \in \mathbb{R}$. The *coadjoint representation* of G on $\mathfrak{g}^*$, the dual space of $\mathfrak{g}$, is the contragredient of the adjoint representation, namely if $\langle u, X\rangle := u(X)$ for $u \in \mathfrak{g}^*$, $X \in \mathfrak{g}$, then

$$\langle (\mathrm{Coad}\, g)u, X\rangle := \langle u, (\mathrm{Ad}\, g^{-1})X\rangle. \tag{2}$$

This defines an action of G on $\mathfrak{g}^*$ and we will write $g \cdot u := (\mathrm{Coad}\, g)u$ to denote this coadjoint action.

Every $X \in \mathfrak{g}$ defines a linear coordinate function ξ_X on $\mathfrak{g}^*$ by $\xi_X(u) := \langle u, X\rangle$. Moreover, from (2), we have $\xi_X(g \cdot u) = \xi_{(\mathrm{Ad}\, g^{-1})X}(u)$.

The 'elementary classical systems' with invariance group G are connected homogeneous symplectic G-manifolds. It is known [13, 14] that these are, up to a covering symplectomorphism, identifiable to orbits of an *affine* action A_θ of G on $\mathfrak{g}^*$. The *linear part* of A_θ is always the coadjoint action of G; classification of these homogeneous G-spaces reduces thus to the classification of the inhomogeneous part θ, which in turn is given by its cohomology class $[\theta]$ in $H^1(\mathrm{G}; \mathfrak{g}^*)$. Note that with each element of $H^1(\mathrm{G}; \mathfrak{g}^*)$ we can associate an element of $H^2(\mathfrak{g}, \mathbb{R})$.

Ideally, we would like to treat with coadjoint actions only, instead of affine actions, since the former are simple to describe and easy to calculate. Martínez-Alonso [6] noticed that the splitting group $\bar{\mathrm{G}}$ could serve that purpose. Let M be a homogeneous symplectic G-space. The group $\bar{\mathrm{G}}$ acts on M via the projection μ through the G-action and so the kernel of $\mu : \bar{\mathrm{G}} \to \mathrm{G}$ acts identically on M; and conversely, the actions of $\bar{\mathrm{G}}$ on M for which $\ker \mu$ acts trivially correspond to actions of G on M. All the symplectic actions of $\bar{\mathrm{G}}$ so considered are Poisson actions and therefore the HSM for G are simply orbits of the coadjoint action of $\bar{\mathrm{G}}$ on which the kernel of $\mu : \bar{\mathrm{G}} \to \mathrm{G}$ acts identically; then the associated action of G on M is transitive (i.e. M is an elementary classical system for G), and conversely.

In short, the classification of 'classical' and 'quantum' actions of a given group has a single unifying principle.

We also recall that the coalgebra $\mathfrak{g}^*$ carries a natural G-invariant *Poisson structure*, which can be defined as follows. For any $u \in \mathfrak{g}^*$, the tangent space $T_u\mathfrak{g}^* \cong \mathfrak{g}^*$, so if $f \in C^\infty(\mathfrak{g}^*)$ we can regard $(\mathrm{d}f)_u : T_u\mathfrak{g}^* \to \mathbb{R}$ as an element $\mathrm{d}f(u)$ of $\mathfrak{g}$. The Poisson bracket is then given by [15]

$$\{f, g\}_{\mathrm{P}}(u) := \langle u, [\mathrm{d}f(u), \mathrm{d}g(u)]\rangle. \tag{3}$$

In particular, since $d\xi_X(u) = X$ for $X \in \mathfrak{g}$, we have

$$\{\xi_X, \xi_Y\}_P(u) = \langle u, [X, Y] \rangle = \xi_{[X,Y]}(u).$$

Writing $x_i = \xi_{X_i}$ where the X_i fill out a basis for $\mathfrak{g}$, we thus have from the Lie algebra commutation relations $[X_i, X_j] = c_{ij}{}^k X_k$:

$$\{f, g\}_P = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \{x_i, x_j\}_P = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \xi_{[X_i, X_j]} = c_{ij}{}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} x_k. \tag{4}$$

It is worth noting that a G-invariant Poisson structure on $\mathfrak{g}^*$ is not necessarily unique. Another such structure is given by

$$\{f, g\}_\beta(u) := \{f, g\}_P(u) - \beta(df(u), dg(u))$$

where β is a 2-cocycle on $\mathfrak{g}$; this is essentially equivalent to (3) only if β is of the form $\beta(X, Y) = \alpha([X, Y])$ for some $\alpha \in \mathfrak{g}^*$, that is, β is a coboundary. As hinted above, the multiplicity of Poisson structures on $\mathfrak{g}^*$ is thus classified by the cohomological properties of $\mathfrak{g}$. We remark that the Lie algebra of the Galilei group admits non-trivial 2-cocycles [16], whereas that of the Poincaré group does not. In the former case, one may recover uniqueness of the invariant Poisson structure on the coalgebra by extending to the eleven-dimensional splitting group of which the Galilei group is a quotient; and the same procedure allows one to obtain projective representations of the Galilei group from linear representations.

The natural symplectic structure on any orbit M of the coadjoint action can be obtained from the Poisson structure. Indeed, the orbits are the symplectic leaves of the natural Poisson structure defined in the coalgebra. If, for each $X \in \mathfrak{g}$, $\tilde{X}$ denotes the 'fundamental vector field' on $\mathfrak{g}^*$ given by

$$(\tilde{X}f)(u) := \frac{d}{dt} f(\exp(-tX) \cdot u)|_{t=0} \tag{5}$$

then $(\tilde{X}\xi_Y)(u) = \langle u, [X, Y] \rangle = \{\xi_X, \xi_Y\} \supset_P(u)$. The fields $\tilde{X}$ are tangent to the orbit M; if $j: M \to \mathfrak{g}^*$ is the inclusion, then the symplectic 2-form ω on M is given [17] by

$$\omega(j_* \tilde{X}, j_* \tilde{Y})(u) := \langle u, [X, Y] \rangle. \tag{6}$$

Here $j_* \tilde{X}, j_* \tilde{Y}$ are the fundamental vector fields of the action of G restricted to M. This symplectic structure is automatically G-invariant. By (6) the Poisson bracket associated with ω on the orbit is simply the restriction of $\{\cdot, \cdot\}_P$. The associated volume form is a G-invariant measure on M (the Liouville measure); after a suitable normalisation, this measure will be denoted by λ_M, or simply by λ, if a fixed orbit M is understood.

2.2. *The Stratonovich–Weyl correspondence*

The coadjoint orbits are connected with irreducible unitary representations of G by the Kirillov orbit method [5, 9]. Suppose then that we are given a connected Lie group G, a linear irreducible unitary representation U of G on a Hilbert space $\mathcal{H}$, and an associated coadjoint orbit $M \subset \mathfrak{g}^*$. For instance, in the case of the Galilei and Poincaré groups, the equivalence classes of projective unitary irreducible representations and the coadjoint orbits which can be physically interpreted as massive particles are identically parametrised. We wish to define a generalisation of the Weyl correspondence [2] which associated an operator A on $\mathcal{H}$ to a (generalised) function W_A on M in a linear one-to-one way; thus, we conjecture the existence of an operator-valued kernel $\Omega: M \to \{\text{operators on } \mathcal{H}\}$ such that $W_A(u) = \mathrm{Tr}[A\Omega(u)]$.

The general requirements for such a kernel were first sketched in a remarkable paper by Stratonovich [18]. We condense them in the following definition.

Definition. Let G be a Lie group and U a linear irreducible unitary representation of G on a Hilbert space $\mathscr{H}$. Let M be a symplectic homogeneous G-space and let λ be a (suitably normalised) G-invariant measure on M. A *Stratonovich-Weyl quantiser* for the triple (G, U, M) is a function $\Omega : M \to \{\text{operators on } \mathscr{H}\}$ which satisfies, for all $u \in M$:

(i) $\Omega(u)$ is self-adjoint (7*a*)

(ii) $Tr[\Omega(u)] = 1$ (7*b*)

(iii) $U(g)\Omega(u)U(g)^{-1} = \Omega(g \cdot u)$ for all $g \in \mathrm{G}$ (7*c*)

(iv) $$\int_M \mathrm{Tr}[\Omega(u)\Omega(v)]\Omega(v)\,\mathrm{d}\lambda(v) = \Omega(u). \tag{7d}$$

The correspondence $A \mapsto W_A$ determined by (G, U, M, Ω) is given by

$$W_A(u) := \mathrm{Tr}[A\Omega(u)]. \tag{8a}$$

We finally require that

(v) $A \mapsto W_A$ be one-to-one.

Let us write, provisionally, $B = \int_M W_A(u)\Omega(u)\,\mathrm{d}\lambda(u)$. Then (7*d*) gives

$$\begin{aligned} W_B(u) &= \mathrm{Tr}[\Omega(u)B] = \int_M W_A(v)\,\mathrm{Tr}[\Omega(u)\Omega(v)]\,\mathrm{d}\lambda(v) \\ &= \mathrm{Tr}\left(A \int_M \Omega(v)\,\mathrm{Tr}[\Omega(u)\Omega(v)]\,\mathrm{d}\lambda(v)\right) = \mathrm{Tr}[A\Omega(u)] = W_A(u) \end{aligned}$$

and thus $B = A$. This gives the *Stratonovich-Weyl correspondence* inverting (8*a*):

$$A = \int_M W_A(u)\Omega(u)\,\mathrm{d}\lambda(u). \tag{8b}$$

It is important to remark that the postulate (7*d*) thus implies that *both* directions (8) of the correspondence $A \rightleftarrows W_A$ are implemented by the *same* kernel $\Omega(u)$. In other words, our quantiser is the 'dequantiser' too.

We have omitted to specify the exact conditions which guarantee convergence and free interchange of the various integrals and traces, preferring at this stage to illuminate the general scheme by physically relevant examples. For instance, for non-compact groups $\Omega(u)$ will not generally be trace-class, but (7*b*) should hold in a weak sense.

Now let us spell out the consequences of (7) for the Stratonovich-Weyl correspondence. Using (8*a*), we obtain at once:

(*a*) A self-adjoint$\Rightarrow W_A$ is real, and $W_{A^+} = \bar{W}_A$ in general

(*b*) W_I is the constant function 1

(*c*) $W_{U(g)AU(g)^{-1}}(g \cdot u) \equiv W_A(u)$.

Furthermore, we have the *tracial property* of the correspondence:

(*d*) $$\int_M W_A(u)W_B(u)\,\mathrm{d}\lambda(u) = \mathrm{Tr}[AB]. \tag{9}$$

Indeed, using (8*a*) and (8*b*), both sides of (9) are equal to $\mathrm{Tr}[A\int_M \Omega(u)W_B(u)\,\mathrm{d}\lambda(u)]$.

If E denotes an appropriate function space on M, then (7d) can be rephrased as

(*e*) $K(u, v) := \mathrm{Tr}[\Omega(u)\Omega(v)]$ is the *reproducing kernel* for E. (10)

The *twisted product* $f \times g$ of two functions f, g in E is defined by

$$(f \times h)(u) = \int_M \int_M L(u, v, w) f(v) h(w)\, \mathrm{d}\lambda(v)\, \mathrm{d}\lambda(w)$$

where $L(u, v, w)$ remains to be determined. By definition, this should correspond to the composition of operators on $\mathcal{H}$, i.e. we require that $W_A \times W_B = W_{AB}$ for any A, B. From (8) we obtain

$$\begin{aligned}(W_A \times W_B)(u) &= W_{AB}(u) = \mathrm{Tr}[\Omega(u)AB] \\ &= \int_M \int_M \mathrm{Tr}[\Omega(u)\Omega(v)\Omega(w)]\, W_A(v)\, W_B(w)\, \mathrm{d}\lambda(v)\, \mathrm{d}\lambda(w)\end{aligned}$$

and we conclude that

(*f*) $L(u, v, w) = \mathrm{Tr}[\Omega(u)\Omega(v)\Omega(w)]$ is the trikernel for the twisted product on M.

Since $W_I = 1$, it follows from (9) that

$$\mathrm{Tr}\, A = \int_M W_A(u)\, \mathrm{d}\lambda(u) \tag{11}$$

which, together with (10), yields the *tracial identity* for the twisted product:

$$\int_M (f \times h)(u)\, \mathrm{d}\lambda(u) = \int_M f(u) h(u)\, \mathrm{d}\lambda(u) \tag{12}$$

on account of

$$\begin{aligned}\int_M (f \times h)(u)\, \mathrm{d}\lambda(u) &= \int_M \int_M \int_M f(v) L(u, v, w) h(w)\, \mathrm{d}\lambda(u)\, \mathrm{d}\lambda(v)\, \mathrm{d}\lambda(w) \\ &= \int_M \int_M \int_M f(v)\, W_{\Omega(v)\Omega(w)}(u) h(w)\, \mathrm{d}\lambda(u)\, \mathrm{d}\lambda(v)\, \mathrm{d}\lambda(w) \\ &= \int_M \int_M f(v) K(v, w) h(w)\, \mathrm{d}\lambda(v)\, \mathrm{d}\lambda(w) \\ &= \int_M f(v) h(v)\, \mathrm{d}\lambda(v).\end{aligned}$$

Comparing (11) and (12) with (1), one sees that this twisted product does indeed comply with the classical rule for computing expected values of observables. It is apparent now that (9) or equivalently (10) or equivalently (12) is the key property for the equivalence of expected values calculated in the Moyal representation and those computed using the formulation in Hilbert space. The twisted product is automatically equivariant:

$$(f \times h)^g(u) = (f^g \times h^g)(u) \qquad \text{for all } g \in \mathrm{G}$$

where $f^g(u) := f(g^{-1} \cdot u)$.

We may now state more precisely what we mean by an 'elementary quantum system'.

Definition. An *elementary quantum system in the Moyal representation* is a homogeneous symplectic G-manifold M, together with an equivariant algebra of functions on M, where the algebra product satisfies the tracial identity (12).

The definition is, in principle, fairly weak, in the sense that it leaves open the possibility of the existence of Moyal quantum systems not arrived at via a Stratonovich–Weyl correspondence. However, as far as we know, the availability of a Moyal representation for an elementary quantum system given in the conventional formulation by a pair (G, U) depends on the existence of a sw quantiser Ω. In section 4, we construct such a quantiser for the Poincaré group.

The question of the uniqueness (up to unitary equivalence) of the quantiser is obviously of importance. We claim at present no general results on uniqueness; however, in all the cases which have so far been examined (the Heisenberg groups, SU(2), and the massive orbits for the Galilei and Poincaré groups), the Stratonovich–Weyl quantiser is essentially unique. It is likely that there is a deep group-theoretic connection here.

2.3. Example 1: pure spin systems

Consider the group $G = SU(2)$, the invariance group for pure spins. Its coadjoint orbits, apart from the origin, are spheres (since SU(2) acts on $g^* \cong \mathbb{R}^3$ by rotations), and its irreducible unitary representations are the well known $\mathscr{D}^j$, for j a half-integer. The integrality conditions select a discrete set of spheres in g^* corresponding to the various $\mathscr{D}^j$. We identify all the spheres for convenience. For this case, the Stratonovich–Weyl quantiser has been determined by two of us [10], following the outline of Stratonovich [18].

Let $\boldsymbol{n}$ be a point on the sphere of radius 1, and let $\Delta^j(\boldsymbol{n}) := \sum_{r,s=-j}^{j} Z^j_{rs}(\boldsymbol{n})|js\rangle\langle jr|$ be the quantiser, with the matrix elements Z^j_{rs} as yet undetermined. The covariance condition (7c) can be written as

$$[m_j(\tilde{R})\Delta^j](\boldsymbol{n}) := \mathscr{D}^j(\tilde{R})\Delta^j(R^{-1}\boldsymbol{n})\mathscr{D}^j(\tilde{R})^{-1} = \Delta^j(\boldsymbol{n})$$

for $\tilde{R} \in SU(2)$, R being the corresponding rotation. The action m_j of SU(2) has an easily determined set of fixed points: one obtains [10] that $\Delta^j = \sum_{l=0}^{2j} \lambda^j_l F^j_l$, where

$$F^j_l(\boldsymbol{n}) := \sum_{r,s=-j}^{j} \left(\frac{2l+1}{2j+1}\right)^{1/2} \langle jls(r-s)|jr\rangle\, Y_{l,r-s}(\boldsymbol{n})|js\rangle\langle jr| \tag{13}$$

where the $\langle jls(r-s)|jr\rangle$ are Clebsch–Gordan coefficients and the $Y_{l,r-s}$ are the usual spherical harmonics on the sphere. (The notation here corrects an error in [10]; in order to obtain correctly formulae (2.22) and (2.23) of that paper, the indices r and s must be permuted.)

The Δ^j are Hermitian matrices (7a) only if the λ^j_l are real. The reproducing kernel for the space of spherical harmonics of degree $\leqslant 2j$ is

$$K^j(\boldsymbol{m}, \boldsymbol{n}) = \frac{4\pi}{2j+1} \sum_{l=0}^{2j} \sum_{m=-l}^{l} Y_{lm}(\boldsymbol{m})\, \bar{Y}_{lm}(\boldsymbol{n})$$

if one adopts, as one must,

$$\mathrm{d}\lambda(\boldsymbol{n}) = \frac{2j+1}{4\pi}\, d\boldsymbol{n} = \frac{2j+1}{4\pi} \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$$

as the Liouville measure. One verified that the F^j_l have the orthogonality property

$$\mathrm{Tr}[F^j_k(\boldsymbol{m})F^j_l(\boldsymbol{n})] = \delta_{kl} \sum_{m=-l}^{l} Y_{lm}(\boldsymbol{m})\bar{Y}_{lm}(\boldsymbol{n}).$$

Now, using (10), the tracial condition (7*d*) gives $(\lambda^j_l)^2 = 4\pi/(2j+1)$. Since $\mathrm{Tr}(F^j_l(\boldsymbol{n})) = 0$ if $l \neq 0$, the condition (7*b*) yields also that $\lambda^j_0 > 0$. Hence we find

$$\lambda^j_0 = \left(\frac{4\pi}{2j+1}\right)^{1/2} \qquad \lambda^j_l = \pm\left(\frac{4\pi}{2j+1}\right)^{1/2} \qquad \text{for } l = 1, 2, \ldots, 2j. \tag{14}$$

The sign ambiguities in (14) are the only measure of non-uniqueness in the sw quantiser Δ^j. Physically, it makes sense to select all signs positive [10]. Thus we finally arrive at

$$\Delta^j(\boldsymbol{n}) = \sum_{r,s=-j}^{j} Z^j_{rs}(\boldsymbol{n})|js\rangle\langle jr| = \sum_{r,s=-j}^{j} \sum_{l=0}^{2j} \frac{[4\pi(2l+1)]^{1/2}}{2j+1} \langle jls(r-s)|jr\rangle Y_{l,r-s}(\boldsymbol{n})|js\rangle\langle jr|. \tag{15}$$

The kernel Δ^j is the Stratonovich–Weyl quantiser for the j spin.

The twisted product of two functions f, g in the space spanned by the matrix elements Z^j_{rs} is given by

$$f \times g(\boldsymbol{n}) = \int_{S^2}\int_{S^2} f(\boldsymbol{m})g(\boldsymbol{k})L(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{k})\,\mathrm{d}\lambda(\boldsymbol{m})\,\mathrm{d}\lambda(\boldsymbol{k})$$

where $L(\boldsymbol{n}, \boldsymbol{m}, \boldsymbol{k}) = \mathrm{Tr}[\Delta^j(\boldsymbol{n})\Delta^j(\boldsymbol{m})\Delta^j(\boldsymbol{k})]$. The functions $Z^j_{rs}(\boldsymbol{n})$ have the orthogonality and product properties:

$$\int_{S^2} Z^j_{rs}(\boldsymbol{n})Z^j_{tu}(\boldsymbol{n})\,\mathrm{d}\lambda(\boldsymbol{n}) = \delta_{ru}\delta_{st} \qquad Z^j_{rs} \times Z^j_{tu} = \delta_{st}Z^j_{ru}$$

as may be verified directly.

We have in particular the *spin eigenstates*:

$$Z^j_{mm}(\boldsymbol{n}) = \sum_{l=0}^{2j} \frac{2l+1}{2j+1}\langle jlm0|jm\rangle P_l(\cos\theta)$$

where the P_l are the Legendre polynomials. If $W_z := \sum^j_{m=-j} mZ^j_{mm}$ is the symbol associated with the J_z spin operator, then

$$W_z(\boldsymbol{n}) = \sum_{m=-j}^{j}\sum_{l=0}^{2j} m\frac{2l+1}{2j+1}\langle jlm0|jm\rangle P_l(\cos\theta) = [j(j+1)]^{1/2}\cos\theta.$$

By means of the quantiser (15), the dynamics of spin was revisited in [10] and Fourier analysis on SU(2) was recast in scalar form. Applications to special function theory are given in [19].

2.4. *Example 2: non-relativistic elementary quantum systems*

One seeks the projective unitary irreducible representations of $\mathrm{G} = \mathbb{R}^4 \rtimes (\mathbb{R}^3 \rtimes \mathrm{SO}(3))$, the identity component of the Galilean group, acting on $\mathbb{R}^4$ by $(b, \boldsymbol{a}, \boldsymbol{v}, R) : (\boldsymbol{x}, t) \mapsto (R\boldsymbol{x} + \boldsymbol{v}t + \boldsymbol{a}, t + b)$. To obtain *linear* representations, one replaces G by its splitting group $\bar{\mathrm{G}}$ [7], which is eleven-dimensional and may be described as follows. Let $\bar{\mathfrak{g}}$ be the Lie algebra generated by $\{H, P^i, J^i, K^i, M\}$ (for $i = 1, 2, 3$) with the commutation relations:

$$[J^i, J^j] = \varepsilon^{ij}{}_k J^k \qquad [J^i, K^j] = \varepsilon^{ij}{}_k K^k \qquad [J^i, P^j] = \varepsilon^{ij}{}_k P^k$$

$$[K^i, H] = P^i \qquad [K^i, P^j] = \delta^{ij} M$$

with all other commutators zero.

Elimination of the central element M gives the usual Galilean commutation relations for $\mathfrak{g}$, so that $\bar{\mathfrak{g}}$ is a central extension of $\mathfrak{g}$; letting $\bar{G}$ be the connected and simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$, we have constructed $\bar{G}$ as a central extension of the covering group $\tilde{G}$ of G by $\mathbb{R}$. Elements of $\bar{G}$ may be written as $(\theta, g) = (\exp(-\theta M), \exp(-bH)\exp(\boldsymbol{a}\cdot\boldsymbol{P} + \boldsymbol{v}\cdot\boldsymbol{K})\tilde{R})$ with $\theta \in \mathbb{R}$ and $g = (b, \boldsymbol{a}, \boldsymbol{v}, \tilde{R}) \in \tilde{G}$. The composition law of $\bar{G}$ obtained from the above commutation relations is

$$(\theta, g)\cdot(\theta', g') = (\theta + \theta' + \omega(g, g'), gg')$$

where $\omega(g, g') = \frac{1}{2}(-b'\boldsymbol{v}\cdot R\boldsymbol{v}' - \boldsymbol{v}\cdot R\boldsymbol{a}' + \boldsymbol{a}\cdot R\boldsymbol{v}')$ is the factor system. As before, $\tilde{R} \in$ SU(2) and R is the SO(3) element by which $\tilde{R}$ acts on $\mathbb{R}^3$.

The unitary irreducible representations of $\bar{G}$ which interest us may be obtained by the induced representation method and act on the momentum space $\mathcal{H}^j = L^2(\mathbb{R}^3, \mathrm{d}^3\boldsymbol{\xi}) \otimes \mathbb{C}^{2j+1}$, where j is a half-integer number, by

$$[U_{muj}(\theta, b, \boldsymbol{a}, \boldsymbol{v}, \tilde{R})\Phi](\xi)$$
$$:= \exp\left(\mathrm{i}\left[\theta + bu + \frac{b|\boldsymbol{\xi}|^2}{2m} - \boldsymbol{\xi}\cdot\boldsymbol{a} - \tfrac{1}{2}m\boldsymbol{a}\cdot\boldsymbol{v}\right]\right)\mathcal{D}^j(\tilde{R})\Phi(R^{-1}(\boldsymbol{\xi} + m\boldsymbol{v})) \tag{16}$$

where $\mathcal{D}^j$ is the unitary irreducible representation of SU(2) on $\mathbb{C}^{2j+1}$.

The coadjoint orbits of $\bar{G}$ have been described in [6] and may be obtained as follows. One first computes the adjoint action of $\bar{G}$ on the generators $H, \boldsymbol{P}, \boldsymbol{J}, \boldsymbol{K}, M$ of $\bar{\mathfrak{g}}$; denoting the coordinates on $\bar{\mathfrak{g}}^*$ by $h = \xi_H, \boldsymbol{p} = \xi_P, \boldsymbol{j} = \xi_J, \boldsymbol{k} = \xi_K, m = \xi_M$, which transform according to $\mathrm{Ad}(g^{-1})$, one finds $(\theta, b, \boldsymbol{a}, \boldsymbol{v}, R)\cdot(h, \boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k}, m)$ explicitly. (We carry out the analogous calculation for the Poincaré group in detail in the next section.) Three invariant quantities appear: m itself, $u = 2mh - |\boldsymbol{p}|^2$ and $|m\boldsymbol{j} + \boldsymbol{p}\times\boldsymbol{k}|^2$; these are the 'Casimir functions' for the canonical Poisson structure of the coalgebra of $\bar{G}$, which are constant on the maximal-dimensional orbits, which thus have dimension 8.

Fix $m > 0$ and $u \in \mathbb{R}$; fix also $s = |\boldsymbol{j} + (1/m)\boldsymbol{p}\times\boldsymbol{k}|$; then one obtains a coadjoint orbit $\mathcal{O}_{mus}$ on which one may introduce coordinates $\boldsymbol{q} := \boldsymbol{k}/m, \boldsymbol{p}$, and if $s > 0$, $\boldsymbol{s} := \boldsymbol{j} + (1/m)\boldsymbol{p}\times\boldsymbol{k}$. The coadjoint action of $\bar{G}$ on $\mathcal{O}_{mus}$ reduces to [6]

$$(\theta, b, \boldsymbol{a}, \boldsymbol{v}, \tilde{R})\cdot(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s}) = \left(R\left(\boldsymbol{q} - \frac{b}{m}\boldsymbol{p}\right) + \boldsymbol{a} + b\boldsymbol{v}, R\boldsymbol{p} - m\boldsymbol{v}, R\boldsymbol{s}\right). \tag{17}$$

It can be checked that (4) reduces to

$$\begin{aligned} &\{q^i, q^j\}_{\mathrm{P}} = \{p^i, p^j\}_{\mathrm{P}} = 0 \qquad &&\{q^i, p^j\}_{\mathrm{P}} = \delta^{ij} \\ &\{q^i, s^j\}_{\mathrm{P}} = \{p^i, s^j\}_{\mathrm{P}} = 0 \qquad &&\{s^i, s^j\}_{\mathrm{P}} = \varepsilon^{ij}{}_k s^k. \end{aligned} \tag{18}$$

From (17) and (18) it is clear that $\mathcal{O}_{mus}$ is isomorphic to $\mathbb{R}^6 \times \boldsymbol{S}^2$ as a symplectic manifold, if $s > 0$, and that $\mathcal{O}_{mu0}$ is symplectically isomorphic to $\mathbb{R}^6$, with canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$. We interpret $\mathcal{O}_{mus}$ as the elementary classical non-relativistic particle of mass m, spin s and internal energy u (for most practical purposes, u can be taken to be zero).

We now take up the question of quantisation. For *massive spinless Galilean particles*, the appropriate triple is $(\bar{G}, U_{m00}, \mathcal{O}_{m00})$ where $m>0$. The desired quantiser is given by the *Grossmann–Royer reflection operators* [11, 20]:

$$[\Pi(\boldsymbol{q},\boldsymbol{p})\Phi](\boldsymbol{\xi}) := 2^3 \exp[2i\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{\xi})]\Phi(2\boldsymbol{p}-\boldsymbol{\xi})$$

acting on $\mathcal{H} = L^2(\mathbb{R}^3, d^3\boldsymbol{\xi})$. These operators are self-adjoint and satisfy

$$\mathrm{Tr}[\Pi(\boldsymbol{q},\boldsymbol{p})] = 1 \qquad \mathrm{Tr}[\Pi(\boldsymbol{q},\boldsymbol{p})\Pi(\boldsymbol{q}',\boldsymbol{p}')] = (2\pi)^3\delta(\boldsymbol{q}-\boldsymbol{q}')\delta(\boldsymbol{p}-\boldsymbol{p}').$$

Moreover, it is readily checked, using (16) and (17), that

$$U_{m00}(g)\Pi(\boldsymbol{q},\boldsymbol{p})U_{m00}(g)^{-1} = \Pi(g\cdot(\boldsymbol{q},\boldsymbol{p})).$$

(In fact, this holds for g belonging to a much larger group than the Galilei group [21].) Thus, if $d\lambda(\boldsymbol{q},\boldsymbol{p}) := (2\pi)^{-3}\,d^3\boldsymbol{q}\,d^3\boldsymbol{p}$, then Π satisfies (7) and so is a quantiser. As was first noticed by Grossmann and Royer, formula (8*b*) with the present Stratonovich–Weyl kernel and Liouville measure, is equivalent to the old Weyl correspondence rule. We recall the formula for the Wigner functions:

$$W_\Phi(\boldsymbol{q},\boldsymbol{p}) := \langle\Phi|\Pi(\boldsymbol{q},\boldsymbol{p})|\Phi\rangle = 2^3\int_{\mathbb{R}^2}\bar{\hat{\Phi}}(\boldsymbol{q}+\boldsymbol{v})\hat{\Phi}(\boldsymbol{q}-\boldsymbol{v})\exp(2i\boldsymbol{p}\cdot\boldsymbol{v})\,d^3\boldsymbol{v} \tag{19}$$

where $\hat{\Phi}$ denotes the Fourier transform.

Finally, in order to quantise *massive Galilean particles with spin*, we may consider the triple $(\bar{G}, U_{m0j}, \mathcal{O}_{m0s})$ with $j>0$ a half-integer. It is shown in [11], and can be verified directly from the preceding paragraphs, that $\Omega_j(\boldsymbol{q},\boldsymbol{p},\boldsymbol{n}) := \Pi(\boldsymbol{q},\boldsymbol{p})\otimes\Delta^j(\boldsymbol{n})$, acting on $\mathcal{H}^j = L^2(\mathbb{R}^3, d^3\xi)\otimes\mathbb{C}^{2j+1}$, satisfies the properties (7), and thus provides a Stratonovich–Weyl quantiser for Galilean spinning particles.

3. Relativistic classical elementary systems

In this section, we describe coadjoint orbits for the Poincaré group $\mathcal{P}$. As usual, M_4 denotes Minkowski space, and if $x=(x^0,\boldsymbol{x})$, $y=(y^0,\boldsymbol{y})$ are 4-vectors in M_4, their Lorentz product is denoted $(xy) = -x^0y^0+\boldsymbol{x}\cdot\boldsymbol{y}$.

The group $\mathcal{P}$ is the semidirect product $T_4\ltimes\mathcal{L}$ where $\mathcal{L}$ is the Lorentz group and

$$(a,\Lambda)\cdot(a',\Lambda') = (a+\Lambda a', \Lambda\Lambda') \qquad \text{for } a\in T_4, \Lambda\in\mathcal{L}.$$

The identity component is $\mathcal{P}_0 = T_4\ltimes\mathcal{L}_+^\uparrow = T_4\ltimes SO_0(3,1)$, the proper orthochronous Poincaré group; and in order to assure that only linear representations need be considered, we will work with its splitting group; this turns out to be just the simply connected double cover $\tilde{\mathcal{P}}_0 = T_4\ltimes SL(2,\mathbb{C})$, which does not have non-trivial extensions.

If $\tilde{\Lambda}\in SL(2,\mathbb{C})$, let Λ be its natural image in $SO_0(3,1)$: let $X := x^0I+\boldsymbol{x}\cdot\boldsymbol{\sigma}$ for $x\in T_4$, where $\boldsymbol{\sigma}=(\sigma^1,\sigma^2,\sigma^3)$ is the set of Pauli matrices in $\mathbb{C}^{2\times2}$, be the corresponding Hermitian matrix in $\mathbb{C}^{2\times2}$; then Λx is the 4-vector corresponding to $\tilde{\Lambda}X\tilde{\Lambda}^\dagger$. The product in $\tilde{\mathcal{P}}_0$ obeys

$$(a,\tilde{\Lambda}')\cdot(a',\tilde{\Lambda}') = (a+\Lambda a', \tilde{\Lambda}\tilde{\Lambda}') \qquad \text{for } a\in T_4, \tilde{\Lambda}\in SL(2,\mathbb{C}). \tag{20}$$

The Lie algebra of $\tilde{\mathcal{P}}_0$ (or of $\mathcal{P}_0$ or $\mathcal{P}$) is generated by ten elements H, P^i, J^i, K^i (for $i=1,2,3$) corresponding, respectively, to time translations, space translations, rotations and pure boosts. Any element of $\tilde{\mathcal{P}}_0$ may be written in the standard form

$$g = \exp(-a^0H+\boldsymbol{a}\cdot\boldsymbol{P})\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K})\exp(\alpha\boldsymbol{m}\cdot\boldsymbol{J}) \tag{21}$$

where $a \in T_4$, $\boldsymbol{n}$ and $\boldsymbol{m}$ are unit 3-vectors, $\zeta \geq 0$ and $0 \leq \alpha \leq 2\pi$, with the convention that $\exp(2\pi \boldsymbol{m} \cdot \boldsymbol{J}) = -I$ in $\mathrm{SL}(2, \mathbb{C})$ for all $\boldsymbol{m}$.

The commutation relations for the generators are

$$\begin{aligned} &[J^i, J^j] = \varepsilon^{ij}{}_k J^k \qquad &&[J^i, K^j] = \varepsilon^{ij}{}_k K^k \qquad &&[J^i, P^j] = \varepsilon^{ij}{}_k P^k \\ &[K^i, K^j] = -\varepsilon^{ij}{}_k J^k \qquad &&[K^i, P^j] = \delta^{ij} H \qquad &&[K^i, H] = P^i \end{aligned} \tag{22}$$

as may be verified directly from (20) together with:

$$\exp(\zeta \boldsymbol{n} \cdot \boldsymbol{K}) = \cosh\tfrac{1}{2}\zeta + \sinh\tfrac{1}{2}\zeta \boldsymbol{n} \cdot \boldsymbol{\sigma}$$

$$\exp(\alpha \boldsymbol{m} \cdot \boldsymbol{J}) = \cos\tfrac{1}{2}\alpha - \mathrm{i}\sin\tfrac{1}{2}\alpha \boldsymbol{m} \cdot \boldsymbol{\sigma}.$$

The adjoint action of $\tilde{\mathscr{P}}_0$ on its Lie algebra $\mathfrak{g}$ may be computed as follows. Writing $(\mathrm{ad}\, X) Y := [X, Y]$ for $X, Y \in \mathfrak{g}$, we have

$$\mathrm{Ad}(\exp X) Y = (e^{\mathrm{ad} X}) Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \ldots. \tag{23}$$

From (23) it is easy to obtain $(\mathrm{Ad}(\exp X)) Y$ whenever $X = -a^0 H, \boldsymbol{a} \cdot \boldsymbol{P}, \alpha \boldsymbol{m} \cdot \boldsymbol{J}$ or $\zeta \boldsymbol{n} \cdot \boldsymbol{K}$, and $Y = H, P^i, J^i$ or K^i. For instance, if $X = \zeta \boldsymbol{n} \cdot \boldsymbol{K}$, $Y = H$, then

$$\begin{aligned} &\mathrm{Ad}(\exp(\zeta \boldsymbol{n} \cdot \boldsymbol{K})) H \\ &\quad = H + \zeta[\boldsymbol{n} \cdot \boldsymbol{K}, H] + \frac{\zeta^2}{2!}[\boldsymbol{n} \cdot \boldsymbol{K}, [\boldsymbol{n} \cdot \boldsymbol{K}, H]] + \frac{\zeta^3}{3!}[\boldsymbol{n} \cdot \boldsymbol{K}, [\boldsymbol{n} \cdot \boldsymbol{K}, [\boldsymbol{n} \cdot \boldsymbol{K}, H]]] + \ldots \\ &\quad = H + \zeta \boldsymbol{n} \cdot \boldsymbol{P} + \frac{\zeta^2}{2!} H + \frac{\zeta^3}{3!} \boldsymbol{n} \cdot \boldsymbol{P} + \ldots = (\cosh \zeta) H + (\sinh \zeta) \boldsymbol{n} \cdot \boldsymbol{P}. \end{aligned}$$

In this way we obtain table 1, which, together with (21), exhibits the adjoint action of $\tilde{\mathscr{P}}_0$ in a fully explicit manner. (We use the notation $\Lambda = R_{\alpha m}$ for the rotation obtained from $\tilde{\Lambda} = \exp(\alpha \boldsymbol{m} \cdot \boldsymbol{J}) \in \mathrm{SU}(2)$.)

The coadjoint action of $\tilde{\mathscr{P}}_0$ on $\mathfrak{g}^*$ can now be derived immediately. Let h be the linear coordinate on $\mathfrak{g}^*$ associated with H, and similarly let p^i, j^i, k^i be the coordinates associated to P^i, J^i, K^i $(i = 1, 2, 3)$. The coadjoint action is given in these coordinates by table 2.

Table 1. The adjoint action $\mathrm{Ad}(\exp X) Y$.

X	$-a^0 H$	$\boldsymbol{a} \cdot \boldsymbol{P}$	$\alpha \boldsymbol{m} \cdot \boldsymbol{J}$	$\zeta \boldsymbol{n} \cdot \boldsymbol{K}$
H	H	H	H	$(\cosh \zeta) H + (\sinh \zeta) \boldsymbol{n} \cdot \boldsymbol{P}$
$\boldsymbol{P}$	$\boldsymbol{P}$	$\boldsymbol{P}$	$R_{\alpha m}^{-1} \boldsymbol{P}$	$\boldsymbol{P} + (\sinh \zeta) \boldsymbol{n} H + (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{P}) \boldsymbol{n}$
$\boldsymbol{J}$	$\boldsymbol{J}$	$\boldsymbol{J} - \boldsymbol{a} \times \boldsymbol{P}$	$R_{\alpha m}^{-1} \boldsymbol{J}$	$(\cosh \zeta) \boldsymbol{J} - (\sinh \zeta) \boldsymbol{n} \times \boldsymbol{K} - (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{J}) \boldsymbol{n}$
$\boldsymbol{K}$	$\boldsymbol{K} + a^0 \boldsymbol{P}$	$\boldsymbol{K} - \boldsymbol{a} H$	$R_{\alpha m}^{-1} \boldsymbol{K}$	$(\cosh \zeta) \boldsymbol{K} + (\sinh \zeta) \boldsymbol{n} \times \boldsymbol{J} - (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{K}) \boldsymbol{n}$

Table 2. The coadjoint action $\mathrm{Coad}(\exp X) y$.

X	$-a^0 H$	$\boldsymbol{a} \cdot \boldsymbol{P}$	$\alpha \boldsymbol{m} \cdot \boldsymbol{J}$	$\zeta \boldsymbol{n} \cdot \boldsymbol{K}$
h	h	h	h	$(\cosh \zeta) h - (\sinh \zeta) \boldsymbol{n} \cdot \boldsymbol{p}$
$\boldsymbol{p}$	$\boldsymbol{p}$	$\boldsymbol{p}$	$R_{\alpha m} \boldsymbol{p}$	$\boldsymbol{p} - (\sinh \zeta(\boldsymbol{n} h + (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{p}) \boldsymbol{n}$
$\boldsymbol{j}$	$\boldsymbol{j}$	$\boldsymbol{j} + \boldsymbol{a} \times \boldsymbol{p}$	$R_{\alpha m} \boldsymbol{j}$	$(\cosh \zeta) \boldsymbol{j} + (\sinh \zeta) \boldsymbol{n} \times \boldsymbol{k} - (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{j}) \boldsymbol{n}$
$\boldsymbol{k}$	$\boldsymbol{k} - a^0 \boldsymbol{p}$	$\boldsymbol{k} + \boldsymbol{a} h$	$R_{\alpha m} \boldsymbol{k}$	$(\cosh \zeta) \boldsymbol{k} - (\sinh \zeta) \boldsymbol{n} \times \boldsymbol{j} - (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{k}) \boldsymbol{n}$

The orbits arise from systems of differential equations $\tilde{X}f=0$, where $\tilde{X}$ is the fundamental vector field (5), and X runs over a basis of $\mathfrak{g}$. Due to the commutation relations (22), this yields an involutive system of differential equations, and the coadjoint orbits are the integral manifolds given by the Stefan-Sussmann generalisation of the Frobenius theorem [14]. Explicitly, for $f=f(h, \boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k})$, we have

$$\boldsymbol{p}\cdot\frac{\partial f}{\partial \boldsymbol{k}}=0 \qquad \boldsymbol{p}\times\frac{\partial f}{\partial \boldsymbol{j}}+h\frac{\partial f}{\partial \boldsymbol{k}}=0$$

$$\boldsymbol{p}\times\frac{\partial f}{\partial \boldsymbol{p}}+\boldsymbol{j}\times\frac{\partial f}{\partial \boldsymbol{j}}+\boldsymbol{k}\times\frac{\partial f}{\partial \boldsymbol{k}}=0$$

$$\boldsymbol{p}\frac{\partial f}{\partial h}+h\frac{\partial f}{\partial \boldsymbol{p}}-\boldsymbol{k}\times\frac{\partial f}{\partial \boldsymbol{j}}+\boldsymbol{j}\times\frac{\partial f}{\partial \boldsymbol{k}}=0.$$

In principle, we have to solve these differential equations to find the orbits. In the present case, this system of equations, while not of constant rank, is generically of rank 8, so that the maximal-dimensional orbits arise as level sets of two 'Casimir functions' C_1, C_2 on $\mathfrak{g}^*$. These Casimir functions are easy to guess from other treatments and to obtain explicitly. Let $p=(h, \boldsymbol{p})$ be the 'energy-momentum' 4-vector and let $w=(w^0, \boldsymbol{w})$ be the Pauli-Lubanski 4-vector given by

$$w^0=\boldsymbol{j}\cdot\boldsymbol{p} \qquad \boldsymbol{w}=\boldsymbol{p}\times\boldsymbol{k}+h\boldsymbol{j}. \tag{24}$$

From table 2 one verifies that w^0 transforms like h and $\boldsymbol{w}$ like $\boldsymbol{p}$ under the coadjoint action; in particular, under $\mathrm{Coad}(\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K}))$:

$$\begin{aligned} w^0 &\mapsto (\cosh\zeta)w^0-(\sinh\zeta)\boldsymbol{n}\cdot\boldsymbol{w} \\ \boldsymbol{w} &\mapsto \boldsymbol{w}-(\sinh\zeta)w^0\boldsymbol{n}+(\cosh\zeta-1)(\boldsymbol{n}\cdot\boldsymbol{w})\boldsymbol{n}. \end{aligned} \tag{25}$$

Thus the Casimir functions we seek are

$$C_1:=(pp)=-h^2+\boldsymbol{p}\cdot\boldsymbol{p}$$

$$C_2:=(ww)=-(\boldsymbol{j}\cdot\boldsymbol{p})^2+|\boldsymbol{p}\times\boldsymbol{k}+h\boldsymbol{j}|^2.$$

The next step is to find a suitable set of coordinates on a particular orbit. We shall now restrict ourselves to orbits for which $C_1<0$, and write $C_1=-m^2$ with $m>0$, in order to deal with massive particles only. (It is clear that the other orbits correspond to the zero-mass and tachyon cases. For a classfication of orbits and an early view on the subject, see [22].) They further subdivide according to whether $h=(m^2+\boldsymbol{p}\cdot\boldsymbol{p})^{1/2}$ or $h=-(m^2+\boldsymbol{p}\cdot\boldsymbol{p})^{1/2}$. For convenience, we restrict for the moment to the positive-energy orbits: $h>0$.

Let H_m^+ denote the forward hyperboloid $(xx)=-m^2$, $x^0>0$, and let $\kappa:=(m,0)$ be its vertex. let L_p denote the Lorentz boost which takes κ to p. We identify L_p with the boost $B(\zeta\boldsymbol{n}):=\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K})$ in $\mathscr{P}_0$, where

$$\cosh\zeta=\frac{h}{m} \qquad \sinh\zeta=\frac{(h^2-m^2)^{1/2}}{m} \qquad \boldsymbol{n}=-\frac{\boldsymbol{p}}{(h^2-m^2)^{1/2}}.$$

By a slight abuse of notation, we also identify L_p with the element

$$\frac{h+m-\boldsymbol{p}\cdot\boldsymbol{\sigma}}{[2m(h+m)]^{1/2}}$$

of $\tilde{\mathscr{P}}_0$. If a is a 4-vector, we thus have

$$L_p a = \left(\frac{ha^0 + \boldsymbol{p}\cdot\boldsymbol{a}}{m}, \boldsymbol{a} + \frac{a^0}{m}\boldsymbol{p} + \frac{\boldsymbol{p}\cdot\boldsymbol{a}}{m(m+h)}\boldsymbol{p}\right). \tag{26}$$

Let us also write $\bar{p}$ for the image of p under spatial reflection: $\bar{p} := (h, -\boldsymbol{p})$. Then, since $L_{\bar{p}}p = \kappa$, it follows from (26) that

$$L_p^{-1} = L_{\bar{p}} \qquad \overline{L_p a} = L_{\bar{p}}\bar{a}.$$

From the definition (24), p and w are orthogonal: $(pw) = 0$, and so

$$0 = (pw) = (L_{\bar{p}}pL_{\bar{p}}w) = (\kappa L_{\bar{p}}w)$$

which means that

$$L_{\bar{p}}w = (0, m\boldsymbol{s}) \tag{27}$$

for some 3-vector $\boldsymbol{s}$. Since

$$C_2 = (ww) = (L_{\bar{p}}wL_{\bar{p}}w) = m^2|\boldsymbol{s}|^2$$

is constant on any orbit, we may interpret $\boldsymbol{s}$ as a 'spin' vector.

From $(0, m\boldsymbol{s}) = L_{\bar{p}}w$, (26) yields

$$\boldsymbol{s} = \frac{\boldsymbol{w}}{m} - \frac{1}{m}\left(\frac{w^0}{m} - \frac{\boldsymbol{p}\cdot\boldsymbol{w}}{m(m+h)}\right)\boldsymbol{p} = \frac{\boldsymbol{w}}{m} - \frac{w^0\boldsymbol{p}}{m(m+h)}. \tag{28}$$

We write, as usual, $s = |\boldsymbol{s}|$ (not a 4-vector!) to denote the spin modulus.

For fixed m and s and positive h, we obtain a single orbit $\mathcal{O}_{ms+}$. If $s > 0$, we may take as coordinates on $\mathcal{O}_{ms+}$ the 'momenta' p^1, p^2, p^3 and two spherical coordinates arising from $\boldsymbol{s}$; three coordinates remain to be determined. A possible choice is q^1, q^2, q^3, where

$$\boldsymbol{q} := \frac{\boldsymbol{k}}{h} - \frac{\boldsymbol{p}\times\boldsymbol{w}}{mh(m+h)} = \frac{\boldsymbol{k}}{h} - \frac{\boldsymbol{p}\times\boldsymbol{s}}{h(m+h)}. \tag{29}$$

The set $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s})$, where $s = |\boldsymbol{s}| > 0$ is fixed, gives a system of eight coordinates on the orbit $\mathcal{O}_{ms+}$. Here $\boldsymbol{p}$ ranges over $\mathbb{R}^3$, $\boldsymbol{s}/s$ over the sphere $\boldsymbol{S}^2$ and, for fixed $\boldsymbol{p}$ and $\boldsymbol{s}$, $\boldsymbol{q}$ ranges over $\mathbb{R}^3$. So the coadjoint orbit $\mathcal{O}_{ms+}$ is homeomorphic to $\mathbb{R}^6 \times \boldsymbol{S}^2$, as in the Galilean case.

The Poisson structure on the relevant submanifold of $\mathfrak{g}^*$ may be obtained from (4) and (22), where we use the basis $\{h, p^i, j^i, k^i: i = 1, 2, 3\}$ of coordinate functions on $\mathfrak{g}^*$. Using (24) and (28), the Poisson brackets for the $\mathcal{O}_{ms+}$ coordinate functions are readily obtained. For instance,

$$\{p^i, p^j\}_{\mathrm{P}} = 0$$

$$\begin{aligned}\{p^i, s^j\}_{\mathrm{P}} &= \frac{1}{m}\{p^i, w^j\}_{\mathrm{P}} - \frac{p^j}{m(m+h)}\{p^i, w^0\}_{\mathrm{P}} \\ &= \frac{1}{m}(\varepsilon^j{}_{kl}p^k\{p^i, k^l\}_{\mathrm{P}} + h\{p^i, j^j\}_{\mathrm{P}}) - \frac{p^jp^k}{m(m+h)}\{p^i, j^k\}_{\mathrm{P}} \\ &= \frac{1}{m}(-\varepsilon^j{}_{ki}p^kh + \varepsilon^{ij}{}_khp^k) - \frac{p^j}{m(m+h)}\varepsilon^{ik}{}_lp^kp^l = 0\end{aligned}$$

$$\begin{aligned}\{p^i, q^j\}_P &= \{p^i, h^{-1}k^j - h^{-1}(m+h)^{-1}\varepsilon^j{}_{kl}p^k s^l\}_P \\ &= \frac{1}{h}\{p^i, k^j\}_P - \frac{\varepsilon^j{}_{kl}}{h(m+h)}(\{p^i, p^k\}_P s^l + p^k\{p^i, s^l\}_P) \\ &= \frac{1}{h}(-\delta^{ij}h) = -\delta^{ij}.\end{aligned}$$

The Poisson brackets of the q^i and s^i are similarly computed; the full results are

$$\begin{aligned}&\{q^i, q^j\}_P = \{p^i, p^j\}_P = 0 \qquad && \{q^i, p^j\}_P = \delta^{ij} \\ &\{q^i, s^j\}_P = \{p^i, s^j\}_P = 0 \qquad && \{s^i, s^j\}_P = \varepsilon^{ij}{}_k s^k.\end{aligned}$$

Thus we see that $\{q^i, p^i\}$ are (part of a set of) canonical coordinates, and that, just as in the Galilean case, $\mathcal{O}_{ms+}$ *is isomorphic to* $\mathbb{R}^6 \times \boldsymbol{S}^2$ *as a symplectic manifold*, if $s > 0$. It follows that $\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}\,\mathrm{d}\boldsymbol{s}$ is a Liouville measure on $\mathcal{O}_{ms+}$. The case $s = 0$ gives a six-dimensional orbit $\mathcal{O}_{m0+}$, isomorphic to $\mathbb{R}^6$.

It is useful to have at hand the expressions of the $\mathfrak{g}^*$ coordinates $(h, \boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k})$ over the orbit in terms of the $\mathcal{O}_{ms+}$ coordinates $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s})$. Inverting (24), (28) and (29) yields

$$\begin{aligned}&w^0 = \boldsymbol{p} \cdot \boldsymbol{s} \qquad && \boldsymbol{w} = m\boldsymbol{s} + \frac{\boldsymbol{p} \cdot \boldsymbol{s}}{m+h}\boldsymbol{p} \\ &\boldsymbol{j} = \boldsymbol{q} \times \boldsymbol{p} + \boldsymbol{s} \qquad && \boldsymbol{k} = h\boldsymbol{q} + \frac{\boldsymbol{p} \times \boldsymbol{s}}{m+h}\end{aligned}$$

as functions on $\mathcal{O}_{ms+}$.

Finally, we could recover from table 2 the expression of the coadjoint action of $\tilde{\mathcal{P}}_0$ on $\mathcal{O}_{ms+}$ in terms of the coordinates $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s})$. From (28) and (29) we readily obtain

$$\begin{aligned}&\exp(-a^0 H) \cdot (\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s}) = \left(\boldsymbol{q} - \frac{a^0}{h}\boldsymbol{p}, \boldsymbol{p}, \boldsymbol{s}\right) \\ &\exp(\boldsymbol{a} \cdot \boldsymbol{P}) \cdot (\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s}) = (\boldsymbol{q} + \boldsymbol{a}, \boldsymbol{p}, \boldsymbol{s}) \\ &\exp(\alpha \boldsymbol{m} \cdot \boldsymbol{J}) \cdot (\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s}) = (R_{\alpha m}\boldsymbol{q}, R_{\alpha m}\boldsymbol{p}, R_{\alpha m}\boldsymbol{s}).\end{aligned} \tag{30}$$

These formulae conform to our 'intuition' as to how a relativistic particle should behave.

The effect of the boost is more cumbersome to express. First we note that it acts on the spin coordinates by a rotation. Indeed, if $\boldsymbol{s} \mapsto \boldsymbol{s}'$, $w \mapsto w'$ under the boost $B = \exp(\zeta\boldsymbol{n} \cdot \boldsymbol{K})$, then (27) gives $(0, m\boldsymbol{s}') = L_{Bp}^{-1}w' = L_{Bp}^{-1}Bw = L_{Bp}^{-1}BL_p(0, m\boldsymbol{s})$, so that $\boldsymbol{s}' = R\boldsymbol{s}$ where $R = L_{Bp}^{-1}BL_p$ is the 'Wigner rotation' corresponding to B and p.

From (25) and (26) we derive the explicit expression

$$\boldsymbol{s}' = (\cos\beta)\boldsymbol{s} + \frac{(1-\cos\beta)(\boldsymbol{n} \times \boldsymbol{p}) \cdot \boldsymbol{s}}{|\boldsymbol{n} \times \boldsymbol{p}|^2}\boldsymbol{n} \times \boldsymbol{p} - \frac{\sin\beta}{|\boldsymbol{n} \times \boldsymbol{p}|}(\boldsymbol{n} \times \boldsymbol{p}) \times \boldsymbol{s}$$

which is a rotation with axis $\boldsymbol{n} \times \boldsymbol{p}$ and angle β, where

$$\cos\beta = 1 - \frac{(\cosh\zeta - 1)|\boldsymbol{n} \times \boldsymbol{p}|^2}{(m+h)(m+h')} \qquad \sin\beta = \frac{-\sinh\zeta(m+h) + (\cosh\zeta - 1)\boldsymbol{n} \cdot \boldsymbol{p}}{(m+h)(m+h')|\boldsymbol{n} \times \boldsymbol{p}|}$$

with $h' := (\cosh\zeta)h - (\sinh\zeta)\boldsymbol{n} \cdot \boldsymbol{p}$, as in table 2. This coincides with the transformation formula derived previously by Sudarshan and Mukunda [23]. Note in particular that, whenever $\boldsymbol{p} = 0$, the Wigner rotation reduces to the identity and $\boldsymbol{s}' = \boldsymbol{s}$.

To see how the boost acts on the position coordinates, let us temporarily suppress s and write $(\boldsymbol{q}, \boldsymbol{p})$ as coordinates for the orbit $\mathcal{O}_{m0+}$. Let us write

$$(\boldsymbol{q}', \boldsymbol{p}') := \exp(\zeta \boldsymbol{n} \cdot \boldsymbol{K}) \cdot (\boldsymbol{q}, \boldsymbol{p}).$$

Then

$$\boldsymbol{p}' = \boldsymbol{p} - (\sinh \zeta) h \boldsymbol{n} + (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{p})\boldsymbol{n}$$

as in table 2; again with $h' := (\cosh \zeta) h - (\sinh \zeta) \boldsymbol{n} \cdot \boldsymbol{p}$, we obtain

$$\boldsymbol{q}' = \boldsymbol{q} + \frac{\sinh \zeta}{h'} (\boldsymbol{n} \cdot \boldsymbol{q})\boldsymbol{p} - \frac{(\cosh \zeta - 1)h}{h'} (\boldsymbol{n} \cdot \boldsymbol{q})\boldsymbol{n}. \tag{31}$$

Formula (31) corresponds to a *covariant* transformation of the position coordinate, in the following precise sense: it gives the *rule of transformation of the initial conditions* (for free motion) on changing from one Lorentz frame (with unprimed coordinates) to another (with primed coordinates). Write

$$\boldsymbol{q}'(t') := \boldsymbol{q}' + \frac{\boldsymbol{p}'}{h'} t' \tag{32}$$

and substitute the preceding formulae for $\boldsymbol{q}'$, $\boldsymbol{p}'$, t' together with

$$t' := t \cosh \zeta - (\sinh \zeta) \boldsymbol{q}(t) \cdot \boldsymbol{n}. \tag{33}$$

Using $\boldsymbol{q}(t) = \boldsymbol{q} + (\boldsymbol{p}/h)t$ to eliminate $\boldsymbol{q}$ in the result, one finally obtains

$$\boldsymbol{q}'(t') = \boldsymbol{q}(t) - t(\sinh \zeta)\boldsymbol{n} + (\cosh \zeta - 1)(\boldsymbol{n} \cdot \boldsymbol{q}(t))\boldsymbol{n} \tag{34}$$

stipulating that $(t, \boldsymbol{q}(t))$ transforms under boosts like a 4-vector. Conversely, starting from (33) and (34) and the free motion conditions, one eliminates t and t' from the formulation and recovers (31). To repeat, this last formula does not relate two different coordinations of the same set of events but rather two simultaneity hyperplanes; and, as such, it is the expression of the Lorentz covariance under boosts in a formulation in which time has been eliminated; canonical transformations can be considered as transformations of the initial conditions of a covariant formulation. The same point has been made convincingly in the field theory context [24]. One could consider the following double bundle:

$$\mathcal{O}_{m0+} \xleftarrow{\pi_1} B \xrightarrow{\pi_2} M_4$$

where the manifold $B = \tilde{\mathcal{P}}_0/\mathrm{SU}(2) \approx \mathbb{R}^7$ has the global coordinates $(t, \boldsymbol{b}, \boldsymbol{p})$ so that $\pi_1(t, \boldsymbol{b}, \boldsymbol{p}) = (\boldsymbol{b} - t\boldsymbol{p}/h, \boldsymbol{p})$ and $\pi_2(t, \boldsymbol{b}, \boldsymbol{p}) = (t, \boldsymbol{b})$. On B the Poincaré group acts in the natural manner; by quotienting over the fibres we recover the formulae of the coadjoint action. Physically, B represents the set of trajectories of free particles with initial conditions given by the points of $\mathcal{O}_{m0+}$.

Now we turn to the general case of non-zero spin. The transformation formula of $\boldsymbol{q}$ under boosts is an involved expression with spin-dependent terms. Thus we seek to replace $\boldsymbol{q}$ by another set of position coordinates (at the price of losing the canonical property, of course). It is better to work at the infinitesimal level: we look for a new set of coordinates $\boldsymbol{x} = (x^1, x^2, x^3)$ on the orbit $\mathcal{O}_{ms+}$ such that

$$\{k^i, x^j\}_{\mathrm{P}} = -x^i v^j \qquad (i, j = 1, 2, 3) \tag{35}$$

where $v^j := \{x^j, h\}_{\mathrm{P}} = p^j/h$. This is but the infinitesimal form of (31). This equality does not hold for $\boldsymbol{x}=\boldsymbol{q}$ when $s \neq 0$. Introduce, however,

$$\boldsymbol{x} := \frac{\boldsymbol{k}}{h} - \frac{\boldsymbol{p}\times\boldsymbol{s}}{mh} = \boldsymbol{q} - \frac{\boldsymbol{p}\times\boldsymbol{s}}{m(m+h)}.$$

Then one easily finds that (35) is verified. A straightforward but tedious computation then allows one to check that $\boldsymbol{x}$ transforms under boosts as desired:

$$\boldsymbol{x}' = \boldsymbol{x} + \frac{\sinh\zeta}{h'}(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{p} - \frac{(\cosh\zeta - 1)h}{h'}(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}.$$

We will use $\boldsymbol{x}$ instead of $\boldsymbol{q}$ for labelling the Ω-kernel in the next section; some simplification is thereby effected. The fortunate fact here is that $\mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{p}\mathrm{d}\boldsymbol{s}$ is still a Liouville measure on $\mathcal{O}_{ms+}$.

We summarise the coadjoint action of $\tilde{\mathcal{P}}_0$ on the orbit $\mathcal{O}_{ms+}$ in table 3.

The existence of the covariant position vector $\boldsymbol{x}$ and its distinction from the canonical position vector $\boldsymbol{q}$ were noticed by Pauri and Prosperi [25] and Bel and Martín [26]. The latter obtained the coadjoint action (without the name) by studying the transformation of the initial conditions for Poicaré-invariant systems of second-order differential equations. The characterisations of [25, 26] are, in principle, only local, whereas ours is global; but this is a moot point here. Neglect of these simple facts has obscured the parallel discussion on relativistic 'position operators'; we return to that question later.

Table 3. The coadjoint action $\mathrm{Coad}(\exp X)(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{s})$.

X	$-a^0H$	$\boldsymbol{a}\cdot\boldsymbol{P}$	$\alpha\boldsymbol{m}\cdot\boldsymbol{J}$	$\zeta\boldsymbol{n}\cdot\boldsymbol{K}$
$\boldsymbol{x}$	$\boldsymbol{x} - \frac{a^0}{h}\boldsymbol{p}$	$\boldsymbol{x}+\boldsymbol{a}$	$R_{\alpha m}\boldsymbol{x}$	$\boldsymbol{x} + \frac{\sinh\zeta}{h'}(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{p} - \frac{(\cosh\zeta-1)h}{h'}(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n}$
$\boldsymbol{p}$	$\boldsymbol{p}$	$\boldsymbol{p}$	$R_{\alpha m}\boldsymbol{p}$	$\boldsymbol{p} - (\sinh\zeta)\boldsymbol{n}h + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n}$
$\boldsymbol{s}$	$\boldsymbol{s}$	$\boldsymbol{s}$	$R_{\alpha m}\boldsymbol{s}$	$(\cos\beta)\boldsymbol{s} + \frac{(1-\cos\beta)[\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{s}]}{\lvert\boldsymbol{n}\times\boldsymbol{p}\rvert^2}\boldsymbol{n}\times\boldsymbol{p} - \frac{\sin\beta}{\lvert\boldsymbol{n}\times\boldsymbol{p}\rvert}(\boldsymbol{n}\times\boldsymbol{p})\times\boldsymbol{s}$

4. Construction of the Stratonovich–Weyl quantiser

The unitary irreducible representations of the Poincaré group are constructed by the induced representation method. Since $\tilde{\mathcal{P}}_0 = T_4 \ltimes \mathrm{SL}(2, \mathbb{C})$, the representation space may be realised as a (multicomponent) function space on an orbit of $\mathrm{SL}(2,\mathbb{C})$ in the dual space of T_4. Again we restrict consideration to orbits H_m^+ corresponding to massive particles of positive energy: $\xi = (\xi^0, \boldsymbol{\xi}) \in H_m^+$ iff $(\xi\xi) = -m^2$ and $\xi^0 > 0$. The corresponding representations are given by [4, 27]

$$[U_{mj+}(a, \tilde{\Lambda})\Phi](\xi) = \exp[-\mathrm{i}(a\xi)]\mathscr{D}^j(L_\xi^{-1}\tilde{\Lambda}L_{\Lambda^{-1}\xi})\Phi(\Lambda^{-1}\xi)$$

where $L_\xi^{-1}\tilde{\Lambda}L_{\Lambda^{-1}\xi} \in \mathrm{SU}(2)$ is a 'Wigner rotation'. In the case $\tilde{\Lambda} = L_p$, we shall denote the Wigner rotation by $R(p, \xi) = L_\xi^{-1}L_pL_{L_p^{-1}\xi}$. Thus j is a half-integer and the representation space is $\mathscr{H}_m^{j,+} = \mathbb{C}^{2j+1} \otimes L^2(H_m^+, \mathrm{d}\mu(\xi))$, where μ is the Lorentz-invariant measure: $\mathrm{d}\mu(\xi) := \mathrm{d}^3\boldsymbol{\xi}/\xi^0$.

The coadjoint orbits corresponding to these representations are $\mathcal{O}_{ms+}$, with the same m and corresponding discrete values of spin. To reduce notational clutter, we fix $m > 0$ and a half-integer j, and write simply $\mathscr{H}^j$, U_j, $\mathcal{O}_j$ rather than $\mathscr{H}_m^{j,+}$, U_{mj+} and $\mathcal{O}^{ms+}$. We will identify the sphere of radius s with the unit sphere if $s>0$, and use coordinates $(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})$ on the orbit $\mathcal{O}_j$.

We are now ready to introduce the Stratonovich-Weyl quantiser satisfying (7). The measure on $\mathcal{O}_j$ will be of the form $\mathrm{d}\lambda_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) := C_j\, \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}\boldsymbol{n} = C_j \mathrm{d}^3\boldsymbol{q}\, \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}\boldsymbol{n}$; the precise value of the constant C_j must be determined in the process.

A suitable family of kets for $\mathcal{H}^j$ is $\{|\xi, r\rangle : \xi \in H_m^+, r=-j, -j+1, \ldots, j\}$, subject to the closure relation:

$$\sum_{r=-j}^{j} \int_{H_m^+} |\xi, r\rangle\langle\xi, r|\, \mathrm{d}\mu(\xi) = I$$

and the corresponding orthogonality properties:

$$\langle \xi, r | \xi', r' \rangle = \xi^0 \delta_{rr'} \delta(\boldsymbol{\xi} - \boldsymbol{\xi}').$$

Traces of operators on $\mathcal{H}^j$ are then computed as

$$\operatorname{Tr} A = \sum_{r=-j}^{j} \int_{H_m^+} \langle \xi, r | A | \xi, r \rangle\, \mathrm{d}\mu(\xi).$$

Before deriving the SW quantiser, it is useful to establish a few notational conventions. We will write, for $p, \xi \in M_4$,

$$\{p\xi\} := \frac{(p\xi)}{(pp)}.$$

In particular, $\{pp\}=1$ and $\{\kappa\xi\} = \xi^0/m$. Moreover, we define the hyperbolic reflection

$$M_p\xi := 2\{p\xi\}p - \xi.$$

M_p is an (improper) Lorentz transformation, $M_p(M_p\xi) = \xi$, $M_p p = p$ and $(pM_p\xi) = (p\xi)$. Moreover, $M_\kappa \xi = \bar{\xi}$. We shall write $M_p\boldsymbol{\xi}$ for the 3-vector component of $M_p\xi$. Furthermore, if Λ is any Lorentz transformation, then $\Lambda M_p \Lambda^{-1} = M_{\Lambda p}$. Finally, note the relation $M_p\xi = L_p L_p \bar{\xi}$.

Theorem. The unique Stratonovich-Weyl quantiser Ω_j for the triple $(\tilde{\mathcal{P}}_0, U_j, \mathcal{O}_j)$ is given by

$$\begin{aligned} [\Omega_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})\Phi](\xi) &:= 2^3\{p\xi\}^{3/2} \exp[\mathrm{i}\boldsymbol{x}\cdot(M_p\boldsymbol{\xi} - \boldsymbol{\xi})] \\ &\quad \times \mathcal{D}^j(R(p, \xi))\Delta^j(\boldsymbol{n})\mathcal{D}^j(R(p, M_p\xi)^{-1})\Phi(M_p\xi) \end{aligned} \tag{36}$$

for $\Phi \in \mathcal{H}^j$.

Proof. We must verify that this definition and no other satifies the properties (7).

First we note that (7c) defines a transitive system of covariance; such a system is determined by specifying an operator $\Omega_j(u_0)$ which commutes with the representatives $U_j(g)$ of the isotropy group of u_0. We take $u_0 = (0, 0, \boldsymbol{n}_0)$. Somewhat more generally, it is necessary and sufficient to establish (7c) for some subgroup which contains the isotropy subgroup of u_0.

To this end, we abbreviate $\Omega_j(\boldsymbol{n}) = \Omega_j(0, 0, \boldsymbol{n})$ and consider the subgroup of $\tilde{\mathcal{P}}_0$ generated by SU(2) and the time translations. We assume $j>0$. For $\tilde{R} \in \mathrm{SU}(2)$, we require Ω_j to satisfy

$$[M_j(\tilde{R})\Omega_j](\boldsymbol{n}) := U_j(\tilde{R})\Omega_j(R^{-1}\boldsymbol{n})U_j(\tilde{R})^{-1} = \Omega_j(\boldsymbol{n}).$$

Since $U_j(\tilde{R}) = \mathscr{D}^j(\tilde{R}) \otimes \lambda(\tilde{R})$, where λ is the left regular representation of SU(2) on $L^2(H_m^+, \mathrm{d}\mu)$, any operator fixed by $M_j(\mathrm{SU}(2))$ is of the form

$$\Omega_j(\boldsymbol{n}) = \sum_{l=0}^{2j} \lambda_l^j F_l^j(\boldsymbol{n}) \otimes P_l^j \tag{37}$$

where $\{F_0^j, \ldots, F_{2j}^j\}$ generate the fixed points (13) of $m_j(\mathrm{SU}(2))$, the λ_l^j are constants given by (14), and P_l^j are operators on $L^2(H_m^+, \mathrm{d}\mu)$ such that

$$\langle R\xi | P_l^j | R\xi' \rangle = \langle \xi | P_l^j | \xi' \rangle \qquad \text{for all } \tilde{R} \in \mathrm{SU}(2).$$

Furthermore, each P_l^j must commute with the representatives of the time translations, so that

$$\langle \xi | P_l^j | \xi' \rangle = \delta(|\boldsymbol{\xi}|^2 - |\boldsymbol{\xi}'|^2) k_l^j(\boldsymbol{\xi}, \boldsymbol{\xi}'). \tag{38}$$

We turn now to the tracial condition ($7d$). We first observe that, on account of ($7c$), we can write

$$\Omega_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) = U(T_x) U(L_p) \Omega_j(\boldsymbol{n}) U(L_p)^{-1} U(T_x)^{-1} \tag{39}$$

where T_x denotes the space translation $\exp(\boldsymbol{x} \cdot \boldsymbol{P})$. Indeed, it suffices to note that

$$(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) = T_x \cdot (0, \boldsymbol{p}, \boldsymbol{n}) = (T_x L_p) \cdot (0, 0, \boldsymbol{n})$$

which follows from table 3.

From (39), the trace $\mathrm{Tr}[\Omega_j(u)\Omega_j(v)]$, where $u = (\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})$ and $v = (\boldsymbol{x}', \boldsymbol{p}', \boldsymbol{n}')$ are any two points of the orbit $\mathscr{O}_j$, simplifies to

$$\begin{aligned}
&\mathrm{Tr}[\Omega_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})\Omega_j(\boldsymbol{x}', \boldsymbol{p}', \boldsymbol{n}')] \\
&\qquad = \mathrm{Tr}[U(T_x)\Omega_j(0, \boldsymbol{p}, \boldsymbol{n}) U(T_{-x}) U(T_{x'}) \Omega_j(0, \boldsymbol{p}', \boldsymbol{n}') U(T_{-x'})] \\
&\qquad = \mathrm{Tr}[\Omega_j(\boldsymbol{x} - \boldsymbol{x}', \boldsymbol{p}, \boldsymbol{n}) \Omega_j(0, \boldsymbol{p}', \boldsymbol{n}')].
\end{aligned} \tag{40}$$

Thus we can take $v = (0, \boldsymbol{p}', \boldsymbol{n}')$ without loss of generality. Since the T_x and the L_p do not commute, no further simplification is possible.

Taking account of (40), the tracial condition can now be written as

$$\mathrm{Tr}[\Omega_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) \Omega_j(0, \boldsymbol{p}', \boldsymbol{n}')] = C^j \delta(\boldsymbol{x}) \delta(\boldsymbol{p} - \boldsymbol{p}') K^j(\boldsymbol{n}, \boldsymbol{n}') \tag{41}$$

where $C^j := (2j+1)/4\pi C_j$. The left-hand side of (41) can be expanded as

$$\begin{aligned}
&\sum_{r,r'=-j}^{j} \int_{H_m^+} \int_{H_m^+} \langle \xi, r | \Omega_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) | \xi', r' \rangle \langle \xi', r' | \Omega_j(0, \boldsymbol{p}', \boldsymbol{n}') | \xi, r \rangle \, \mathrm{d}\mu(\xi) \, \mathrm{d}\mu(\xi') \\
&\qquad = \sum_{r,r'=-j}^{j} \int_{H_m^+} \int_{H_m^+} \exp[\mathrm{i}\boldsymbol{x} \cdot (\boldsymbol{\xi} - \boldsymbol{\xi}')] \langle \xi, r | U(L_p) \Omega_j(\boldsymbol{n}) U(L_{\bar{p}}) | \xi', r' \rangle \\
&\qquad\qquad \times \langle \xi', r' | U(L_{p'}) \Omega_j(\boldsymbol{n}') U(L_{\bar{p}'}) | \xi, r \rangle \, \mathrm{d}\mu(\xi) \, \mathrm{d}\mu(\xi').
\end{aligned}$$

Using (37) and the fact that $U(L_{\bar{p}})|\xi, r\rangle = \sum_{s=-j}^{j} \mathscr{D}_{sr}^j(R(p, L_p\xi)) | L_{\bar{p}}\xi, s\rangle$, the integrand is a sum of terms of the form $\langle L_{\bar{p}}\xi | P_l^j | L_{\bar{p}}\xi' \rangle \langle L_{\bar{p}'}\xi' | P_{l'}^j | L_{\bar{p}'}\xi \rangle$. From (26) we note that if $|\boldsymbol{\xi}'| = |\boldsymbol{\xi}|$ and $p \neq \kappa$, then $|L_p\boldsymbol{\xi}'| = |L_p\boldsymbol{\xi}|$ only if $\boldsymbol{\xi}' = \pm\boldsymbol{\xi}$. Since the right-hand side of (41) is not zero, we conclude that (38) may be refined to

$$\langle \xi | P_l^j | \xi' \rangle = f_l^j(\xi^0) \delta(\boldsymbol{\xi} + \boldsymbol{\xi}') + g_l^j(\xi^0) \delta(\boldsymbol{\xi} - \boldsymbol{\xi}')$$

or alternatively

$$P_l^j = f_l^j(\xi^0) P + g_l^j(\xi^0) I$$

with P denoting the parity operator $|\xi\rangle\mapsto|\bar{\xi}\rangle$. The normalisation condition (7*b*), together with $\operatorname{Tr} I=+\infty$, demands that $g^j_l(\xi^0)=0$. Hence

$$\Omega_j(\boldsymbol{n})=\sum_{l=0}^{2j}\lambda^j_l F^j_l(\boldsymbol{n})\otimes 2^3 f^j_l(\xi^0)P$$

where we have renormalised the f^j_l for convenience; by (7*a*), each f^j_l must be a real function.

Thus we can write

$$\begin{aligned}[\Omega_j(\boldsymbol{x},\boldsymbol{p},\boldsymbol{n})\Phi](\xi)&=[U(T_x)U(L_p)\Omega_j(\boldsymbol{n})U(L_{\bar{p}})U(T_{-x})\Phi](\xi)\\&=2^3\sum_{l=0}^{2j}\exp[\mathrm{i}\boldsymbol{x}\cdot(M_p\boldsymbol{\xi}-\boldsymbol{\xi})]\lambda^j_l f^j_l(m\{p\xi\})\mathscr{D}^j(R(p,\xi))\\&\quad\times F^j_l(\boldsymbol{n})\mathscr{D}^j(R(\bar{p},L_p\bar{\xi}))\Phi(M_p\xi).\end{aligned}\tag{42}$$

Since $R(\bar{p},L_p\bar{\xi})=R(p,M_p\xi)^{-1}$, we conclude that

$$\begin{aligned}&C^j\delta(\boldsymbol{x})\delta(\boldsymbol{p}-\boldsymbol{p}')K^j(\boldsymbol{n},\boldsymbol{n}')\\&\quad=\sum_{r=-j}^{j}\langle\xi,r|\Omega(\boldsymbol{x},\boldsymbol{p},\boldsymbol{n})\Omega(0,\boldsymbol{p}',\boldsymbol{n}')|\xi,r\rangle\,\mathrm{d}\mu(\xi)\\&\quad=2^6\int_{H^+_m}\sum_{k,l=0}^{2j}\lambda^j_k\lambda^j_l f^j_k(m\{p\xi\})f^j_l(m\{p'\xi\})\exp[\mathrm{i}\boldsymbol{x}\cdot(M_p\boldsymbol{\xi}-\boldsymbol{\xi})]\langle M_p\xi|M_{p'}\xi\rangle\\&\qquad\times\operatorname{Tr}[\mathscr{D}^j(R(p,\xi))F^j_k(\boldsymbol{n})\mathscr{D}^j(R(p,M_p\xi)^{-1})\\&\qquad\times\{\mathscr{D}^j(R(p',\xi))F^j_l(\boldsymbol{n}')\mathscr{D}^j(R(p',M_{p'}\xi)^{-1})\}^{\dagger}]\,\mathrm{d}\mu(\xi).\end{aligned}\tag{43}$$

Now $\langle M_p\xi|M_{p'}\xi\rangle=(M_p\xi)^0\delta(2\{p\xi\}\boldsymbol{p}-2\{p'\xi\}\boldsymbol{p}')=2^{-3}h\{p\xi\}^{-2}\delta(\boldsymbol{p}-\boldsymbol{p}')$, and so (43) reduces to

$$\begin{aligned}&C^j\delta(\boldsymbol{x})K^j(\boldsymbol{n},\boldsymbol{n}')\\&\quad=\frac{4\pi}{2j+1}\int_{H^+_m}2^3h\{p\xi\}^{-2}\sum_{l=0}^{2j}[f^j_l(m\{p\xi\})]^2\\&\qquad\times\sum_{m=-l}^{l}Y_{lm}(\boldsymbol{n})\bar{Y}_{lm}(\boldsymbol{n}')\exp[\mathrm{i}\boldsymbol{x}\cdot(M_p\boldsymbol{\xi}-\boldsymbol{\xi})]\,\mathrm{d}\mu(\xi).\end{aligned}$$

Thus the functions $(f^j_l)^2$ coincide for $l=0,1,\ldots,2j$. Since

$$\int_{\mathbb{R}^3}2^3\frac{h}{\xi^0}\{p\xi\}\exp[\mathrm{i}\boldsymbol{x}\cdot(M_p\boldsymbol{\xi}-\boldsymbol{\xi})]\,\mathrm{d}^3\boldsymbol{\xi}=\int_{\mathbb{R}^3}\exp[\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{\eta}]\,\mathrm{d}^3\boldsymbol{\eta}=(2\pi)^3\delta(\boldsymbol{x})$$

and since (7*b*) now gives $f^j_0(m)=1$, we conclude that $f^j_l(\xi^0)=(\xi^0/m)^{3/2}$ for all l and that $C^j=(2\pi)^3$ for all j. Substituting $f^j_l(m\{p\xi\})=\{p\xi\}^{3/2}$ in (42) gives the desired result (36).

It remains to check that (7*a*) holds; by the established covariance, we need only check that $\Omega_j(\boldsymbol{n})$ is self-adjoint. Since $[\Omega_j(\boldsymbol{n})\Phi](\xi)=2^3(\xi^0/m)^{3/2}\Delta^j(\boldsymbol{n})\Phi(\bar{\xi})$, it is clear from the invariance of the measure $\mathrm{d}\mu(\xi)$ under $\xi\mapsto\bar{\xi}$ that $\Omega_j(\boldsymbol{n})$ is symmetric. It has moreover a bounded inverse, and hence is a self-adjoint operator.

The Liouville measure on $\mathcal{O}_j$ should thus be normalised to

$$\mathrm{d}\lambda(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) = \frac{\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{2j+1}{4\pi}\,\mathrm{d}\boldsymbol{n} \qquad \text{for } j>0.$$

For the case $j=0$, where $\boldsymbol{x} = \boldsymbol{q}$, the proof is similar (indeed simpler) and the corresponding Liouville measure is just $\mathrm{d}\lambda(\boldsymbol{q}, \boldsymbol{p}) := (2\pi)^{-3}\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}$.

Remark. The SW quantiser is given uniquely by the formula (36), but there is the residual ambiguity in the SU(2) quantiser Δ^j, as the signs of the $\lambda_1^j, \ldots, \lambda_{2j}^j$ may be chosen freely in (14). We will keep our choice of only positive λ_l^j and will continue to speak, with a slight abuse of language, of a 'unique' Stratonovich–Weyl quantiser in the Poincaré group case.

5. The phase-space formalism for Klein–Gordon particles

There is a remarkable dichotomy in relativistic quantum theory between the presentation of elementary systems by means of unitary irreducible representations of the Poincaré group and their presentation by means of covariant 'wave' equations of various sorts. The objects associated with the latter were historically introduced first, and are easier to handle because of their manifest covariance properties; also, they lend themselves to the introduction of interactions. On the other hand, the theory of unitary irreducible representations, introduced by Wigner [4] for reasons of principle, treats all the particles in a unified way, allowing a more systematic classification.

In practice, this state of affairs indicates that we have by no means finished our task. The relation between the 'Wigner' and 'covariant' approaches is by now well understood. One must adapt the Stratonovich–Weyl construct to the more usual context of wave equations; of course, the new families of Stratonovich–Weyl operators will be essentially equivalent to that given by (36). We carry out this adaptation in section 7 for the case of spin-half particles; meanwhile, in order to gain familiarity with the phase-space formalism, it will be useful to treat the case of spinless particles, where the dichotomy becomes vacuous. In this case the representation space becomes $L^2(H_m^+, \mathrm{d}\mu(\xi))$, which may be immediately identified with the momentum wavefunction space.

The $(\boldsymbol{q}, \boldsymbol{p})$ labelling of observables and states can be considered phase-space coordinates for any Lorentzian observer; by construction the formalism is invariant, although not in a manifest way.

Now suppose that we have a state prepared at $t=0$ in the configuration $\rho_0(\boldsymbol{q}, \boldsymbol{p}) = \rho_0(u)$. Its free evolution is given by the classical formula:

$$\begin{aligned}
\rho_t(\boldsymbol{q}, \boldsymbol{p}) &= \mathrm{Tr}\left[U(T_t)^{-1}(2\pi)^{-3}\left(\int_{\mathbb{R}^6}\rho_0(\boldsymbol{q}', \boldsymbol{p}')\Omega_0(\boldsymbol{q}', \boldsymbol{p}')\,\mathrm{d}^3\boldsymbol{q}'\,\mathrm{d}^3\boldsymbol{p}'\right)U(T_t)\Omega_0(\boldsymbol{q}, \boldsymbol{p})\right] \\
&= \mathrm{Tr}\left[(2\pi)^{-3}\int_{\mathbb{R}^6}\rho_0(\boldsymbol{q}', \boldsymbol{p}')\Omega_0(\boldsymbol{q}', \boldsymbol{p}')\Omega_0(\boldsymbol{q} - t\boldsymbol{p}/h, \boldsymbol{p})\,\mathrm{d}^3\boldsymbol{q}'\,\mathrm{d}^3\boldsymbol{p}'\right] \\
&= \int_{\mathbb{R}^6}\rho_0(\boldsymbol{q}', \boldsymbol{p}')\delta(\boldsymbol{q} - t\boldsymbol{p}/h - \boldsymbol{q}')\delta(\boldsymbol{p} - \boldsymbol{p}')\,\mathrm{d}^3\boldsymbol{q}'\,\mathrm{d}^3\boldsymbol{p}' \\
&= \rho_0(\boldsymbol{q} - t\boldsymbol{p}/h, \boldsymbol{p}) = \rho_0(\boldsymbol{q} - \boldsymbol{v}t, \boldsymbol{p})
\end{aligned} \tag{44}$$

with $\boldsymbol{v}$ denoting the velocity $\boldsymbol{p}/h$; and the expected value of the observable f in the state ρ is

$$\langle f\rangle_{\rho_t} = (2\pi)^{-3}\int_{\mathbb{R}^6} \rho_t(\boldsymbol{q},\boldsymbol{p}) f(\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}.$$

As $\int \rho_t(\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p} = (2\pi)^3$, this is the promised (1)!

The inherent simplicity of these fomulae lies in the fact that there is no need to 'quantise' f, i.e. to express it in operatorial form, a hopeless task in general. In this sense, phase space quantisation is an *ab initio* quantisation.

Not every configuration ρ_0 qualifies as a state, however. The more important states are the *Wigner functions*:

$$\begin{aligned} W_\Phi(\boldsymbol{q},\boldsymbol{p}) &:= \langle\Phi|\Omega_0(\boldsymbol{q},\boldsymbol{p})|\Phi\rangle \\ &= 2^3\int_{\mathbb{R}^3} \{p\xi\}^{3/2}\exp[\mathrm{i}\boldsymbol{q}\cdot(M_p\boldsymbol{\xi}-\boldsymbol{\xi})]\bar{\Phi}(\xi)\Phi(M_p\xi)\,\mathrm{d}^3\boldsymbol{\xi}/\xi^0. \end{aligned} \tag{45}$$

Note that $W_\Phi \times W_\Phi = W_\Phi$. Thus it is clear from (12) that we have

$$(2\pi)^{-3}\int_{\mathbb{R}^6} W_\Phi(\boldsymbol{q},\boldsymbol{p})(f\times\bar{f})(\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p} = (2\pi)^{-3}\int_{\mathbb{R}^6} |W_\Phi\times f|^2(\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p} \geqslant 0$$

so W_Φ qualifies as a state. The simplest example is the 'plane wave' $\Phi_k(\boldsymbol{\xi}) = k^0\delta(\boldsymbol{\xi}-\boldsymbol{k})$. Substituting in (45) we get

$$\begin{aligned} W_{\Phi_k}(\boldsymbol{q},\boldsymbol{p}) &= 2^3\int_{\mathbb{R}^3}\{p\xi\}^{3/2}\exp[\mathrm{i}\boldsymbol{q}\cdot(M_p\boldsymbol{\xi}-\boldsymbol{\xi})]\delta(\boldsymbol{\xi}-\boldsymbol{k})\delta(M_p\boldsymbol{\xi}-\boldsymbol{k})(k^0)^2/\xi^0\,\mathrm{d}^3\boldsymbol{\xi} \\ &= 2^3\int_{\mathbb{R}^3}\{p\xi\}^{3/2}\exp[2\mathrm{i}\boldsymbol{q}\cdot(\{p\xi\}\boldsymbol{p}-\boldsymbol{\xi})]\delta(\boldsymbol{\xi}-\boldsymbol{k})\delta(2(\boldsymbol{p}-\boldsymbol{k}))(k^0)^2/\xi^0\,\mathrm{d}^3\boldsymbol{\xi} \\ &= k^0\delta(\boldsymbol{p}-\boldsymbol{k}). \end{aligned}$$

Let us denote generally by $\mathrm{Op}(f)$ the operator corresponding by (8b) to a function f; by definition, $\mathrm{Op}(W_A) = A$.

Now we prove that $(2\pi)^{-3}\int \boldsymbol{q}\Omega_0(\boldsymbol{q},\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p}$ is the Newton–Wigner operator $\boldsymbol{Q}_{\mathrm{op}}$; that is, $\mathrm{Op}(\boldsymbol{q}) = \boldsymbol{Q}_{\mathrm{op}}$. This one can expect from the proof by O'Connell and Wigner [28] that $\boldsymbol{Q}_{\mathrm{op}}$ is unique in fulfilling $\langle\boldsymbol{Q}_{\mathrm{op}}\rangle_\Phi(t) = \langle\boldsymbol{Q}_{\mathrm{op}}\rangle_\Phi(0) + t\langle H_{\mathrm{op}}^{-1}\boldsymbol{P}_{\mathrm{op}}\rangle_\Phi$, similarly to our (32).

We have

$$\begin{aligned} &(2\pi)^{-3}\int \boldsymbol{q}\langle\Phi|\Omega_0(\boldsymbol{q},\boldsymbol{p})|\Phi\rangle\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p} \\ &\quad = \pi^{-3}\int_{\mathbb{R}^9}\{p\xi\}^{3/2}\boldsymbol{q}\exp[\mathrm{i}\boldsymbol{q}\cdot(M_p\boldsymbol{\xi}-\boldsymbol{\xi})]\bar{\Phi}(\xi)\Phi(M_p\xi)(\xi^0)^{-1}\,\mathrm{d}^3\boldsymbol{\xi}\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{p} \\ &\quad = (2\pi)^{-3}\int_{\mathbb{R}^9} q\exp[\mathrm{i}\boldsymbol{q}\cdot(\boldsymbol{v}-\boldsymbol{\xi})]\bar{\Phi}(\xi)\Phi(v) \\ &\qquad \times\left[\frac{v^0+\xi^0}{2\xi^0v^0}\left(\frac{1+\{v\xi\}}{2}\right)^{-3/4}\right]\mathrm{d}^3\boldsymbol{\xi}\,\mathrm{d}^3\boldsymbol{q}\,\mathrm{d}^3\boldsymbol{v} \end{aligned} \tag{46}$$

where we have made the change of variable $\boldsymbol{p}\mapsto\boldsymbol{v} := M_p\boldsymbol{\xi}$.

Introducing new variables $\boldsymbol{y} := \frac{1}{2}(\boldsymbol{v}+\boldsymbol{\xi})$, $z := \boldsymbol{v}-\boldsymbol{\xi}$, let $h(y, z)$ be the bracketed expression in the integrand (46) in terms of y, z. Due to this expression's symmetry in $\boldsymbol{v}$ and $\boldsymbol{\xi}$, we have $\partial h/\partial z|_{z=0} = 0$. Integrating now over the $\boldsymbol{q}$ variables, we obtain

$$(2\pi)^{-3} \int \boldsymbol{q}\langle\Phi|\Omega(\boldsymbol{q},\boldsymbol{p})|\Phi\rangle \, d^3\boldsymbol{q} \, d^3\boldsymbol{p}$$

$$= -i \int_{\mathbb{R}^6} \frac{\partial}{\partial z}(\delta(z))\bar{\Phi}(y-\tfrac{1}{2}z)\Phi(y+\tfrac{1}{2}z)h(y,z) \, d^3z \, d^3y$$

$$= i \int_{\mathbb{R}^6} \frac{\partial}{\partial z}(\bar{\Phi}(y-\tfrac{1}{2}z)\Phi(y+\tfrac{1}{2}z)h(y,z))|_{z=0} \, d^3y$$

$$= \int_{\mathbb{R}^3} \bar{\Phi}(y)\left(i\frac{\partial}{\partial y} - \frac{iy}{2(y^0)^2}\right)\Phi(y)\frac{d^3y}{y^0}.$$

From our remarks at the end of section 3, it follows that the Newton–Wigner operator for spinless particles corresponds to a covariant position observable. This is not made altogether clear in the original paper by Newton and Wigner [29].

Other relevant observables are associated with the generators of the representation:

$$H_{\rm op} = \xi^0 \qquad \boldsymbol{P}_{\rm op} = \boldsymbol{\xi} \qquad \boldsymbol{J}_{\rm op} = i\frac{\partial}{\partial\boldsymbol{\xi}} \times \boldsymbol{\xi} \qquad \boldsymbol{K}_{\rm op} = i\xi^0\frac{\partial}{\partial\boldsymbol{\xi}}.$$

Similar routine calculations allow one to check

$$\langle\Phi|H_{\rm op}|\Phi\rangle = (2\pi)^{-3}\int_{\mathbb{R}^6} h W_\Phi(\boldsymbol{q},\boldsymbol{p}) \, d^3\boldsymbol{q} \, d^3\boldsymbol{p}$$

$$\langle\Phi|\boldsymbol{P}_{\rm op}|\Phi\rangle = (2\pi)^{-3}\int_{\mathbb{R}^6} \boldsymbol{p} W_\Phi(\boldsymbol{q},\boldsymbol{p}) \, d^3\boldsymbol{q} \, d^3\boldsymbol{p}$$

$$\langle\Phi|\boldsymbol{J}_{\rm op}|\Phi\rangle = (2\pi)^{-3}\int_{\mathbb{R}^6} \boldsymbol{j} W_\Phi(\boldsymbol{q},\boldsymbol{p}) \, d^3\boldsymbol{q} \, d^3\boldsymbol{p}$$

$$\langle\Phi|\boldsymbol{K}_{\rm op}|\Phi\rangle = (2\pi)^{-3}\int_{\mathbb{R}^6} \boldsymbol{k} W_\Phi(\boldsymbol{q},\boldsymbol{p}) \, d^3\boldsymbol{q} \, d^3\boldsymbol{p}.$$

One of the typical properties of the ordinary Wigner function is that its marginals give the correct probabilities for finding the particle localised at a given point or with a given momentum. In the non-relativistic case, these properties come for free from our basic postulates of traciality and covariance. But here we do not have an *a priori* reason for expecting that to happen, nor is it necessary for a consistent phase-space theory.

In fact, we do have

$$(2\pi)^{-3}\int_{\mathbb{R}^3} W_\Phi(\boldsymbol{q},\boldsymbol{p}) \, d^3\boldsymbol{q} = \frac{|\Phi(\boldsymbol{p})|^2}{h}$$

so the marginal with respect to $\boldsymbol{q}$ indeed gives the standard probability of finding a particle with the given momentum. But the analogous property is not true for the integral over $\boldsymbol{p}$.

The standard formulation of the probability for locating the position of a particle is somewhat involved, so it may be worth recalling it. Let $\phi(\boldsymbol{q}, t)$ be the spacetime wavefunction corresponding to Φ:

$$\phi(\boldsymbol{q}, t) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \exp[-\mathrm{i}(\xi^0 t - \boldsymbol{\xi} \cdot \boldsymbol{q})]\Phi(\xi) \, \mathrm{d}^3\boldsymbol{\xi}/\xi^0.$$

By definition, it satisfies the Klein-Gordon equation:

$$\left(-\frac{\partial^2 \phi}{\partial t^2} + \Delta - m^2 \right) \phi = 0.$$

Now, letting

$$\chi := H_{\mathrm{KG}}^{1/2}\phi := (-\Delta + m^2)^{1/4}\phi$$

and $d_{\mathrm{KG}}(\boldsymbol{q}, t) := |\chi(\boldsymbol{q}, t)|^2$, one can check that d_{KG} has the properties required to be interpreted as a probability density in configuration space:

$$\int_{\mathbb{R}^3} d_{\mathrm{KG}}(\boldsymbol{q}, t) \, \mathrm{d}^3\boldsymbol{q} = 1 \qquad \int_{\mathbb{R}^3} \boldsymbol{q} d_{\mathrm{KG}}(\boldsymbol{q}, t) \, \mathrm{d}^3\boldsymbol{q} = \langle \boldsymbol{Q}_{\mathrm{op}} \rangle (t).$$

The trouble with this d_{KG} (as with its higher-spin counterparts) lies in the non-local properties pointed out by Hegerfeldt [30].

On the other hand, we can define a *different* density function from the Wigner function W_Φ corresponding to Φ:

$$d(\boldsymbol{q}, t) := (2\pi)^{-3} \int_{\mathbb{R}^3} W_\Phi(\boldsymbol{q} - t\boldsymbol{p}/h, \boldsymbol{p}) \, \mathrm{d}^3\boldsymbol{p}.$$

It is immediately clear, from the facts already proved, that

$$\int_{\mathbb{R}^3} d(\boldsymbol{q}, t) \, \mathrm{d}^3\boldsymbol{q} = 1 \qquad \int_{\mathbb{R}^3} \boldsymbol{q} \, d(\boldsymbol{q}, t) \, \mathrm{d}^3\boldsymbol{q} = \langle \boldsymbol{Q}_{\mathrm{op}} \rangle (t)$$

but it follows from (44) that d is local (that is, its support does not grow or shrink supraluminally).

The higher moments of d_{KG} and d are of course different. In fact, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} (q^1)^{n_1}(q^2)^{n_2}(q^3)^{n_3} d_{\mathrm{KG}}(\boldsymbol{q}, t) \, \mathrm{d}^3\boldsymbol{q} \\
&\qquad = \int_{\mathbb{R}^3} (q^1)^{\times n_1}(q^2)^{\times n_2}(q^3)^{\times n_3} d(\boldsymbol{q}, t) \, \mathrm{d}^3\boldsymbol{q} \\
&\qquad = \langle (Q^1_{\mathrm{op}})^{n_1}(Q^2_{\mathrm{op}})^{n_2}(Q^3_{\mathrm{op}})^{n_3} \rangle (t)
\end{aligned} \tag{47}$$

where $(q^i)^{\times n_i}$ denotes the n_i-fold twisted product $q^i \times q^i \times \ldots \times q^i$. The fact that $d_{\mathrm{KG}} \neq d$ is related to $(q^i)^{\times n} \neq (q^i)^n$, whereas, as we prove below, $(p^i)^{\times n} = (p^i)^n$. We do not know, however, whether d is always non-negative.

We argue that the rule expressed by (47) is a natural one. In effect, it is universally true that

$$(2\pi)^{-3} \int_{\mathbb{R}^6} f^\times(W_A)(\boldsymbol{q}, \boldsymbol{p}) W_\Phi(\boldsymbol{q}, \boldsymbol{p}) \, \mathrm{d}^3\boldsymbol{q} \, \mathrm{d}^3\boldsymbol{p} = \langle f(A) \rangle_\Phi$$

for an operator A, provided $f^\times$ is the *twisted* function corresponding to f. In the ordinary non-relativistic Moyal mechanics, as long as W_A corresponds to a canonical coordinate, $f^\times$ and f coincide. For examples of non-trivial twisted function computations with the harmonic oscillator Hamiltonian and other important operators see [19].

We turn our attention then to the twisted product. Its trikernel is given by

$$\begin{aligned}
L(\boldsymbol{q}_1, \boldsymbol{p}_1; \boldsymbol{q}_2, \boldsymbol{p}_2; \boldsymbol{q}_3, \boldsymbol{p}_3) &= \mathrm{Tr}[\Omega_0(\boldsymbol{q}_1, \boldsymbol{p}_1)\Omega_0(\boldsymbol{q}_2, \boldsymbol{p}_2)\Omega_0(\boldsymbol{q}_3, \boldsymbol{p}_3)] \\
&= 2^9 \int_{\mathbb{R}^{12}} \{p_1\xi_1\}^{3/2}\{p_2\xi_2\}^{3/2}\{p_3\xi_3\}^{3/2} \\
&\quad \times \exp\{\mathrm{i}[\boldsymbol{q}_1 \cdot (m_{p_1}\xi_1 - \boldsymbol{\xi}_1) + \boldsymbol{q}_2 \cdot (M_{p_2}\boldsymbol{\xi}_2 - \boldsymbol{\xi}_2) + \boldsymbol{q}_3 \cdot (M_{p_3}\boldsymbol{\xi}_3 - \boldsymbol{\xi}_3)]\} \\
&\quad \times \delta(\boldsymbol{\xi} - \boldsymbol{\xi}_1)\delta(M_{p_1}\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\delta(M_{p_2}\boldsymbol{\xi}_2 - \boldsymbol{\xi}_3)\delta(M_{p_3}\boldsymbol{\xi}_3 - \boldsymbol{\xi})\, \mathrm{d}^3\boldsymbol{\xi}_1\, \mathrm{d}^3\boldsymbol{\xi}_2\, \mathrm{d}^3\boldsymbol{\xi}_3\, \mathrm{d}^3\boldsymbol{\xi} \\
&= 2^9 \int_{\mathbb{R}^3} \{p_1\xi\}^{3/2}\{p_2 M_{p_1}\xi\}^{3/2}\{p_3 M_{p_2} M_{p_1}\xi\}^{3/2} \\
&\quad \times \exp\{\mathrm{i}[q_1 \cdot (m_{p_1}\boldsymbol{\xi} - \boldsymbol{\xi}) + \boldsymbol{q}_2 \cdot (M_{p_2}M_{p_1}\boldsymbol{\xi} - M_{p_1}\boldsymbol{\xi}) + \boldsymbol{q}_3 \cdot (\boldsymbol{\xi} - M_{p_2}M_{p_1}\boldsymbol{\xi})]\} \\
&\quad \times \delta(M_{p_3}M_{p_2}M_{p_1}\boldsymbol{\xi} - \boldsymbol{\xi})\, \mathrm{d}^3\boldsymbol{\xi}.
\end{aligned} \tag{48}$$

The equation $M_{p_3}M_{p_2}M_{p_1}\xi = \xi$ has the unique solution:

$$\xi = A(\{p_2p_3\}p_1 - \{p_3p_1\}p_2 + \{p_1p_2\}p_3)$$

where

$$A = (\{p_1p_2\}^2 + \{p_2p_3\}^2 + \{p_3p_1\}^2 - 2\{p_1p_2\}\{p_2p_3\}\{p_3p_1\})^{-1/2}.$$

For simplicity, we abbreviate $a = \{p_2p_3\}$, $b = \{p_3p_1\}$, $c = \{p_1p_2\}$. Substituting in (48), we obtain, after a laborious computation,

$$\begin{aligned}
&L(\boldsymbol{q}_1, \boldsymbol{p}_1; \boldsymbol{q}_2, \boldsymbol{p}_2; \boldsymbol{q}_3, \boldsymbol{p}_3) \\
&\qquad = 2^6 A^{13/2}(abc)^{3/2} \exp(2\mathrm{i}A[b\boldsymbol{q}_1 \cdot \boldsymbol{p}_2 + c\boldsymbol{q}_2 \cdot \boldsymbol{p}_3 + a\boldsymbol{q}_3 \cdot \boldsymbol{p}_1 \\
&\qquad\quad - c\boldsymbol{q}_1 \cdot \boldsymbol{p}_3 - a\boldsymbol{q}_2 \cdot \boldsymbol{p}_1 - b\boldsymbol{q}_3 \cdot \boldsymbol{p}_2]).
\end{aligned}$$

Using (30) and (31), one verifies directly that this trikernel has the desired equivariance property:

$$L(g \cdot u, g \cdot v, g \cdot w) = L(u, v, w) \qquad \text{for } g \in \mathrm{G},\ u, v, w \in \mathcal{C}.$$

In the non-relativistic limit, $a, b, c, A \to 1$. Thus we recover the trikernel for the non-relativistic twisted product:

$$L(\boldsymbol{q}_1, \boldsymbol{p}_1; \boldsymbol{q}_2, \boldsymbol{p}_2; \boldsymbol{q}_3, \boldsymbol{p}_3) = 2^6 \exp[2\mathrm{i}(\boldsymbol{q}_1 \cdot \boldsymbol{p}_2 + \boldsymbol{q}_2 \cdot \boldsymbol{p}_3 + \boldsymbol{q}_3 \cdot \boldsymbol{p}_1 - \boldsymbol{q}_1 \cdot \boldsymbol{p}_3 - \boldsymbol{q}_2 \cdot \boldsymbol{p}_1 - \boldsymbol{q}_3 \cdot \boldsymbol{p}_2)].$$

Let us illustrate how twisted products may be computed by two simple examples. Let $f(\boldsymbol{p})$ be a function of $\boldsymbol{p}$ alone. Then if $h(\boldsymbol{q}, \boldsymbol{p}) = p^j \times f(\boldsymbol{p})$, we find that

$$\begin{aligned}
h(\boldsymbol{q}_1, \boldsymbol{p}_1) &= \pi^{-6} \int_{\mathbb{R}^{12}} (abc)^{3/2} A^{13/2} p_2^j f(\boldsymbol{p}_3) \exp\{-2\mathrm{i}A[a\boldsymbol{p}_1 \cdot (q_2 - \boldsymbol{q}_3) \\
&\quad + b\boldsymbol{p}_2 \cdot (\boldsymbol{q}_3 - \boldsymbol{q}_1) + c\boldsymbol{p}_3 \cdot (\boldsymbol{q}_1 - \boldsymbol{q}_2)]\}\, \mathrm{d}^3\boldsymbol{q}_2\, \mathrm{d}^3\boldsymbol{p}_2\, \mathrm{d}^3\boldsymbol{q}_3\, \mathrm{d}^3\boldsymbol{p}_3 \\
&= \pi^{-3} \int_{\mathbb{R}^9} (abc)^{3/2} A^{13/2} p_2^j f(\boldsymbol{p}_3) \delta(b\boldsymbol{p}_2 - a\boldsymbol{p}_1) \\
&\quad \times \exp\{-2\mathrm{i}A[(a\boldsymbol{p}_1 - c\boldsymbol{p}_3) \cdot \boldsymbol{q}_2 - (b\boldsymbol{p}_2 - c\boldsymbol{p}_3) \cdot \boldsymbol{q}_1]\}\, \mathrm{d}^3\boldsymbol{q}_2\, \mathrm{d}^3\boldsymbol{p}_2\, \mathrm{d}^3\boldsymbol{p}_3
\end{aligned}$$

$$= \pi^{-3} \int_{\mathbb{R}^9} p_2^j f(\boldsymbol{p}_3) \frac{bh_1}{h_3} \delta(\boldsymbol{p}_2 - \boldsymbol{p}_1)$$

$$\times \exp[-2\mathrm{i}(b\boldsymbol{p}_1 - \boldsymbol{p}_3) \cdot (\boldsymbol{q}_2 - \boldsymbol{q}_1)] \, \mathrm{d}^3\boldsymbol{q}_2 \, \mathrm{d}^3\boldsymbol{p}_2 \, \mathrm{d}^3\boldsymbol{p}_3$$

$$= \pi^{-3} p_1^j \int_{\mathbb{R}^6} f(\boldsymbol{p}_3) \frac{bh_1}{h_3} \exp[-2\mathrm{i}(b\boldsymbol{p}_1 - \boldsymbol{p}_3) \cdot (\boldsymbol{q}_2 - \boldsymbol{q}_1)] \, \mathrm{d}^3\boldsymbol{q}_2 \, \mathrm{d}^3\boldsymbol{p}_3$$

$$= p_1^j \int_{\mathbb{R}^3} f(\boldsymbol{p}_3) \frac{bh_1}{h_3} \delta(b\boldsymbol{p}_1 - \boldsymbol{p}_3) \exp[-2\mathrm{i}\boldsymbol{q}_1 \cdot (b\boldsymbol{p}_1 - \boldsymbol{p}_3)] \, \mathrm{d}^3\boldsymbol{p}_3$$

$$= p_1^j \int_{\mathbb{R}^3} f(\boldsymbol{p}_3) \delta(\boldsymbol{p}_3 - \boldsymbol{p}_1) \exp[-2\mathrm{i}\boldsymbol{q}_1 \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_3)] \, \mathrm{d}^3\boldsymbol{p}_3$$

$$= p_1^j f(\boldsymbol{p}_1)$$

as expected. Taking $f(\boldsymbol{p}) = (p^j)^{n-1}$ confirms our previous assertion that $(p^j)^{\times n} = (p^j)^n$.

Furthermore, if $k(\boldsymbol{q}, \boldsymbol{p}) = q^j \times f(\boldsymbol{p})$, a similar calculation gives

$$k(\boldsymbol{q}_1, \boldsymbol{p}_1) = \pi^{-3} \int_{\mathbb{R}^6} q_2^j f(\boldsymbol{p}_3) \frac{bh_1}{h_3} \exp[-2\mathrm{i}(b\boldsymbol{p}_1 - \boldsymbol{p}_3) \cdot (\boldsymbol{q}_2 - \boldsymbol{q}_1)] \, \mathrm{d}^3\boldsymbol{q}_2 \, \mathrm{d}^3\boldsymbol{p}_3$$

$$= \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^3} f(\boldsymbol{p}_3) \frac{bh_1}{h_3} \partial_j \delta(b\boldsymbol{p}_1 - \boldsymbol{p}_3) \exp[2\mathrm{i}\boldsymbol{q}_1 \cdot (b\boldsymbol{p}_1 - \boldsymbol{p}_3)] \, \mathrm{d}^3\boldsymbol{p}_3$$

$$= \frac{1}{2\mathrm{i}} \int_{\mathbb{R}^3} f(\boldsymbol{p}_3(\boldsymbol{r})) \partial_j \delta(\boldsymbol{r}) \exp(-2\mathrm{i}\boldsymbol{q}_1 \cdot \boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}$$

$$= \frac{\mathrm{i}}{2} \frac{\partial}{\partial r^j} [f(\boldsymbol{p}_3(\boldsymbol{r})) \exp(-2\mathrm{i}\boldsymbol{q}_1 \cdot \boldsymbol{r})]|_{\boldsymbol{r}=0}$$

$$= q_1^j f(\boldsymbol{p}_1) + \tfrac{1}{2}\mathrm{i}\partial_j f(\boldsymbol{p}_1)$$

using the change of variable $\boldsymbol{r} = \boldsymbol{p}_3 - b\boldsymbol{p}_1$ and noting that $\boldsymbol{r} = 0$ only when $\boldsymbol{p}_3 = \boldsymbol{p}_1$. We may summarise these results as

$$p^j \times f(\boldsymbol{p}) = p^j f(\boldsymbol{p}) \qquad q^j \times f(\boldsymbol{p}) = q^j f(\boldsymbol{p}) + \frac{\mathrm{i}}{2} \frac{\partial f}{\partial p^j}(\boldsymbol{p}).$$

In an analogous manner it is easy to show that

$$f(\boldsymbol{p}) \times q^j = q^j f(\boldsymbol{p}) - \frac{\mathrm{i}}{2} \frac{\partial f}{\partial p^j}(\boldsymbol{p})$$

so that the *Moyal bracket* of q^j and $f(\boldsymbol{p})$, defined by

$$\{q^j, f(\boldsymbol{p})\}_M := \frac{1}{\mathrm{i}} (q^j \times f(\boldsymbol{p}) - f(\boldsymbol{p}) \times q^j)$$

satisfies

$$\{q^j, f(\boldsymbol{p})\}_M = \frac{\partial f}{\partial p^j}(\boldsymbol{p}) = \{q^j, f(\boldsymbol{p})\}_{\mathrm{P}}.$$

6. Quantised observables

We now 'quantise' the main observables in the Wigner realisation for $j > 0$. As before, we write $H_{\text{op}}, \boldsymbol{P}_{\text{op}}, \boldsymbol{J}_{\text{op}}, \boldsymbol{K}_{\text{op}}, \boldsymbol{Q}_{\text{op}}, \boldsymbol{X}_{\text{op}}$ for the operators corresponding to the coordinates $h, \boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{q}, \boldsymbol{x}$.

First we check that $\text{Op}(f(\boldsymbol{p})) = f(\boldsymbol{P}_{\text{op}})$, where $\boldsymbol{P}_{\text{op}}$ is of course the multiplication operator $\boldsymbol{\xi}$. To simplify the notation, summation over repeated indices of the Wigner spinors is to be understood:

$$\langle\Phi|f(\boldsymbol{p})\Omega_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})|\Psi\rangle$$

$$= 2^3 \int \bar{\Phi}(\xi)\{p\xi\}^{3/2} \exp[\mathrm{i}\boldsymbol{x} \cdot (M_p\boldsymbol{\xi} - \boldsymbol{\xi})] f(\boldsymbol{p}) \mathscr{D}^j(R(p, \xi))\Delta^j(n)$$

$$\times \mathscr{D}^j(R(p, M_p\xi)^{-1})\Psi(M_p\xi) \frac{\mathrm{d}^3\boldsymbol{\xi}}{\xi^0} \frac{2j+1}{4\pi} \mathrm{d}\boldsymbol{n} \frac{\mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}$$

$$= 2^3 \int \bar{\Phi}(\xi)\{p\xi\}^{3/2} \delta(2\boldsymbol{p} - 2\boldsymbol{\xi}) f(\boldsymbol{p}) \mathscr{D}^j(R(p, \xi))\Delta^j(\boldsymbol{n})$$

$$\times \mathscr{D}^j(R(M_p\xi, p))\Psi(M_p\xi) \frac{\mathrm{d}^3\boldsymbol{\xi}}{\xi^0} \frac{2j+1}{4\pi} \mathrm{d}\boldsymbol{n}\, \mathrm{d}^3\boldsymbol{p}$$

$$= \int \bar{\Phi}(\xi) f(\boldsymbol{\xi}) \Phi(\xi)\, \mathrm{d}\mu(\xi) = \langle\Phi|f(\boldsymbol{P}_{\text{op}})|\Psi\rangle.$$

It can be seen that the calculations made in [10] for pure functions of spin are essentially unchanged here and we obtain

$$\text{Op}([j(j+1)]^{1/2}\boldsymbol{n}) = \boldsymbol{S}_{\text{op}}$$

where $\boldsymbol{S}_{\text{op}}$ is the angular momentum generator for spin j. Moreover, it is obvious that if we quantise $[j(j+1)]^{1/2}\boldsymbol{n}f(\boldsymbol{p})$ we get $\boldsymbol{S}_{\text{op}}f(\boldsymbol{P}_{\text{op}}) = f(\boldsymbol{P}_{\text{op}})\boldsymbol{S}_{\text{op}}$.

Now we compute $\boldsymbol{X}_{\text{op}}$:

$$\langle\Phi|\boldsymbol{x}\Omega_j(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})|\Psi\rangle$$

$$= 2^3 \int \bar{\Phi}(\xi)\{p\xi\}^{3/2}\boldsymbol{x} \exp[\mathrm{i}\boldsymbol{x} \cdot (M_p\boldsymbol{\xi} - \boldsymbol{\xi})] f(\boldsymbol{p}) \mathscr{D}^j(R(p, \xi))\Delta^j(\boldsymbol{n})$$

$$\times \mathscr{D}^j(R(M_p\xi, p))\Psi(M_p\xi) \frac{\mathrm{d}^3\boldsymbol{\xi}}{\xi^0} \frac{2j+1}{4\pi} \mathrm{d}\boldsymbol{n} \frac{\mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}. \tag{49}$$

We now change variables as in (46): first $\boldsymbol{p} \mapsto \boldsymbol{v} := M_p\boldsymbol{\xi}$, then $\boldsymbol{y} := \frac{1}{2}(\boldsymbol{v} + \boldsymbol{\xi})$, $\boldsymbol{z} := \boldsymbol{v} - \boldsymbol{\xi}$. Integrating over the $\boldsymbol{q}$ and $\boldsymbol{n}$ variables and then integrating by parts, (49) yields

$$\mathrm{i} \int_{\mathbb{R}^3} \frac{\partial}{\partial z} [\bar{\Phi}(y - \tfrac{1}{2}z)\Psi(y + \tfrac{1}{2}z) h(y, z) \mathscr{D}^j(R'(y, z)) \mathscr{D}^j(R''(y, z))]|_{z=0}\, \mathrm{d}^3 y$$

where $h(y, z)$ is the same function as before and R', R'' denote, respectively, $R(p, \xi)$ and $R(M_p\xi, p)$ as functions of y, z. The explicit expression for $R(p, \xi)$ is

$$R(p, \xi) = \frac{(\xi^0 + m)(h + m) - \boldsymbol{p} \cdot \boldsymbol{\xi} - \mathrm{i}\boldsymbol{\sigma} \cdot (\boldsymbol{\xi} \times \boldsymbol{p})}{[2(\xi^0 + m)(h + m)(m^2 - (\xi p))]^{1/2}}.$$

We have then for $p \approx \xi$:

$$R(p, \xi) \approx R(M_p \xi, p) \approx 1 - \frac{\mathrm{i}}{4} \frac{\boldsymbol{\sigma} \cdot (\boldsymbol{\xi} \times \boldsymbol{v})}{m(\xi^0 + m)}$$

and

$$\mathscr{D}^j(R'(y, z)) \approx \mathscr{D}^j(R''(y, z)) \approx 1 - \frac{\mathrm{i}}{2} \frac{(\boldsymbol{S}_{\mathrm{op}} \times \boldsymbol{y}) \cdot \boldsymbol{z}}{m(y^0 + m)}.$$

We conclude that

$$\boldsymbol{X}_{\mathrm{op}} = \mathrm{i}\left(\frac{\partial}{\partial \boldsymbol{\xi}} - \frac{\boldsymbol{\xi}}{2(\xi^0)^2}\right) + \frac{\boldsymbol{S}_{\mathrm{op}} \times \boldsymbol{\xi}}{m(\xi^0 + m)}.$$

Now $(\boldsymbol{S}_{\mathrm{op}} \times \boldsymbol{\xi})/m(\xi^0 + m)$ is the 'quantisation' of $(\boldsymbol{s} \times \boldsymbol{p})/m(h+m)$, where $\boldsymbol{s} = [j(j+1)]^{1/2} \boldsymbol{n}$. It follows that

$$\boldsymbol{Q}_{\mathrm{op}} = \mathrm{i}\left(\frac{\partial}{\partial \boldsymbol{\xi}} - \frac{\boldsymbol{\xi}}{2(\xi^0)^2}\right)$$

for all j: the Newton–Wigner operator has a form which is independent of j (in the Wigner realisation). This often forgotten fact has been recalled recently by Chakrabarti [31].

Routine calculations now establish

$$\boldsymbol{J}_{\mathrm{op}} = \mathrm{i} \frac{\partial}{\partial \boldsymbol{\xi}} \times \boldsymbol{\xi} + \boldsymbol{S}_{\mathrm{op}} \qquad \boldsymbol{K}_{\mathrm{op}} = \mathrm{i}\xi^0 \frac{\partial}{\partial \boldsymbol{\xi}} + \frac{\boldsymbol{\xi} \times \boldsymbol{S}_{\mathrm{op}}}{\xi^0 + m}.$$

7. The phase-space formalism for Dirac particles

We begin by fixing some notation. In the spirit of [32], we think of 4-vectors $y = (y^0, \boldsymbol{y})$ as the corresponding matrices $y^0 I + \boldsymbol{y} \cdot \boldsymbol{\sigma}$, and freely multiply them: $y_1 y_2 = (y_1^0 y_2^0 + \boldsymbol{y}_1 \cdot \boldsymbol{y}_2, y_1^0 \boldsymbol{y}_2 + y_2^0 \boldsymbol{y}_1 + \mathrm{i}\boldsymbol{y}_1 \times \boldsymbol{y}_2)$. This leads to a remarkable simplification in several formulae. As

$$\left(\frac{(\xi^0 + m, \boldsymbol{\xi})}{[2(\xi^0 + m)]^{1/2}}\right)^2 = \xi$$

we shall write simply

$$\xi^{1/2} := \frac{(\xi^0 + m, \boldsymbol{\xi})}{[2(\xi^0 + m)]^{1/2}}.$$

(We shall not have occasion to use the other square roots of ξ.) Our Dirac matrices are

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

i.e. we choose the chiral representation. Define

$$\not{\xi} := -(\xi\gamma) = \begin{pmatrix} 0 & \xi \\ \bar{\xi} & 0 \end{pmatrix}.$$

It is well known and easy to check that, if $D^{i,j}$ denotes the standard finite-dimensional representations of SL(2, $\mathbb{C}$) and

$$S(\tilde{\Lambda}) := \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & \tilde{\Lambda}^{+-1} \end{pmatrix} = \begin{pmatrix} D^{1/2,0}(\tilde{\Lambda}) & 0 \\ 0 & D^{0,1/2}(\tilde{\Lambda}) \end{pmatrix}$$

with $\tilde\Lambda \in \mathrm{SL}(2, \mathbb{C})$, then

$$S(\tilde\Lambda)\not{\xi}S(\tilde\Lambda)^{-1} = \Lambda\not{\xi}.$$

The method of relating Dirac's and Wigner's realisations for spin-half particles is also well known. We shall proceed from the latter to the former. Consider the representation space $\mathcal{H}_m^{1/2,+} := \mathbb{C}^2 \otimes L^2(H_m^+, \mathrm{d}\mu(\xi))$. We shall introduce a Hilbert space $\hat{\mathcal{H}}_m^{1/2,+}$ of 4-spinors ismorphic to $\mathcal{H}_m^{1/2,+}$. If $\Phi \in \mathcal{H}_m^{1/2,+}$, define

$$v_A(\xi) := \frac{1}{\sqrt{2}} D^{1/2,0}(L_\xi)\Phi(\xi) = \left(\frac{\xi}{2m}\right)^{1/2} \Phi(\xi)$$

$$v_B(\xi) := \frac{1}{\sqrt{2}} D^{0,1/2}(L_\xi)\Phi(\xi) = \left(\frac{\bar\xi}{2m}\right)^{1/2} \Phi(\xi).$$

We have then

$$v_B(\xi) = \frac{\bar\xi}{m} v_A(\xi) \qquad v_A(\xi) = \frac{\xi}{m} v_B(\xi). \tag{50}$$

If $\Phi, \Phi' \in \mathcal{H}_m^{1/2,+}$, we abbreviate $\bar\Phi(\xi) \cdot \Phi'(\xi) = \bar\Phi_1(\xi)\Phi_1'(\xi) + \bar\Phi_2(\xi)\Phi_2'(\xi)$. Then their inner product is given by

$$\begin{aligned}\langle\Phi|\Phi'\rangle &= \int_{H_m^+} \bar\Phi(\xi) \cdot \Phi'(\xi)\, \mathrm{d}\mu(\xi) = 2\int_{H_m^+} \bar v_A(\xi) \cdot v_B'(\xi)\, \mathrm{d}\mu(\xi) \\ &= 2\int_{H_m^+} \bar v_B(\xi) \cdot v_A'(\xi)\, \mathrm{d}\mu(\xi) \\ &= \int_{H_m^+} (\bar v_A(\xi) \cdot v_B'(\xi) + \bar v_B(\xi) \cdot v_A'(\xi))\, \mathrm{d}\mu(\xi).\end{aligned}$$

If we now set

$$\Psi(\xi) := \begin{pmatrix} v_A(\xi) \\ v_B(\xi) \end{pmatrix} = \frac{1}{\sqrt{2}} (D^{1/2,0} \oplus D^{0,1/2})(L_\xi) \begin{pmatrix} \Phi(\xi) \\ \Phi(\xi) \end{pmatrix} \tag{51}$$

we get

$$(\Psi|\Psi') := \int_{H_m^+} \bar\Psi(\xi)\gamma^0\Psi'(\xi)\, \mathrm{d}\mu(\xi) = \langle\Phi|\Phi'\rangle.$$

(Again we sum over the repeated indices of the spinors.) The Hilbert space $\hat{\mathcal{H}}_m^{1/2,+}$ of 4-spinors of the form (51), with this inner product, is then unitarily equivalent to $\mathcal{H}_m^{1/2,+}$.

Now, $\hat{\mathcal{H}}_m^{1/2,+}$ is the space of solutions of the Dirac equation. In effect, from (50) we see that

$$m\begin{pmatrix} v_A(\xi) \\ v_B(\xi) \end{pmatrix} = \begin{pmatrix} 0 & \xi \\ \bar\xi & 0 \end{pmatrix} \begin{pmatrix} v_A(\xi) \\ v_B(\xi) \end{pmatrix}$$

that is, $(\not{\xi} - m)\Psi = 0$, the Dirac equation in momentum space.

The 'basis' $\{\Phi_{k,\pm 1/2}(\xi)\} = \{k^0\delta(\boldsymbol{\xi} - \boldsymbol{k})\binom{1}{0}, k^0\delta(\boldsymbol{\xi} - \boldsymbol{k})\binom{0}{1} \colon \boldsymbol{k} \in \mathbb{R}^3\}$ transforms under the given isomorphism $T\colon \mathcal{H}_m^{1/2,+} \to \hat{\mathcal{H}}_m^{1/2,+}$ into

$$\{\Psi_{k,\pm 1/2}(\xi)\} := \left\{ k^0\delta(\boldsymbol{\xi} - \boldsymbol{k}) \begin{pmatrix} (k/2m)^{1/2}\binom{1}{0} \\ (\bar k/2m)^{1/2}\binom{1}{0} \end{pmatrix}, k^0\delta(\boldsymbol{\xi} - \boldsymbol{k}) \begin{pmatrix} (k/2m)^{1/2}\binom{0}{1} \\ (\bar k/2m)^{1/2}\binom{0}{1} \end{pmatrix} \colon \boldsymbol{k} \in \mathbb{R}^3 \right\}$$

and so $(\Psi_{k,r}|\Psi_{k',r'}) = k^0\delta(\boldsymbol{k} - \boldsymbol{k}')\delta_{rr'}$. One naturally uses this basis to compute traces in $\hat{\mathcal{H}}_m^{1/2,+}$.

On $\hat{\mathcal{H}}_m^{1/2,+}$ we consider the representation $V_D := TU_{1/2}T^{-1}$. Explicitly

$$[V_D(a, \tilde\Lambda)\Psi](\xi) = \exp[-\mathrm{i}(a\xi)]S(\tilde\Lambda)\Psi(\Lambda^{-1}\xi). \tag{52}$$

We have recovered the usual expression for the relativistic invariance of the Dirac equation, in the chiral representation. (Of course, we can make a similarity transformation to recover any other representation we like.) Note that (52) can be considered as a (reducible) representation on the whole space of 4-spinors; thus defined, V_D commutes with the projector $P_D = (\not{\xi} + m)/2m$ on the subspace $\hat{\mathcal{H}}_m^{1/2,+}$, and preserves the form of spinors given by (50) and (51).

A well known result in representation theory establishes that a representation leaves invariant an inner product given by an associated Hermitian matrix (here γ^0) if and only if it is equivalent to its contragredient conjugate representation. This is the reason for choosing the representation space of $D^{1/2,0} \oplus D^{0,1/2}$, which is reduced to $\hat{\mathcal{H}}_m^{1/2,+}$ by our definitions.

We can now define the Stratonovich–Weyl quantiser for the (positive-energy) Dirac particles $\Omega_D(x, p, n)$ as $\Omega_D(u) := T\Omega_{1/2}(u)T^{-1}$. We compute this by recalling that $D^{1/2,0}(\tilde{R}) = D^{0,1/2}(\tilde{R}) = \mathscr{D}^{1/2}(\tilde{R})$ for $\tilde{R} \in \mathrm{SU}(2)$:

$$\begin{aligned}
&[\Omega_D(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})\Psi](\xi) \\
&\qquad = 2^3\{p\xi\}^{3/2} \exp[\mathrm{i}\boldsymbol{x} \cdot (M_p\boldsymbol{\xi} - \boldsymbol{\xi})] S(L_\xi) S(L_\xi^{-1} L_p L_{L_p^{-1}\xi}) \Delta_D(\boldsymbol{n}) \\
&\qquad\quad \times S(L_{L_p^{-1}\xi}^{-1} L_p^{-1} L_{M_p\xi}) S(L_{M_p\xi}^{-1}) \Psi(M_p\xi) \\
&\qquad = 2^3\{p\xi\}^{3/2} \exp[\mathrm{i}\boldsymbol{x} \cdot (M_p\boldsymbol{\xi} - \boldsymbol{\xi})] S(L_p L_{L_p^{-1}\xi}) \Delta_D(\boldsymbol{n}) S(L_{L_p^{-1}\xi} L_p^{-1}) \Psi(M_p\xi)
\end{aligned} \tag{53a}$$

where $\Delta_D(\boldsymbol{n}) := (\Delta^{1/2} \oplus \Delta^{1/2})(\boldsymbol{n})$. In particular

$$[\Omega_D(0, 0, \boldsymbol{n})\Psi](\xi) = 2^3\left(\frac{\xi^0}{m}\right)^{3/2} S(L_\xi)\Delta_D(\boldsymbol{n})S(L_\xi)\Psi(\xi). \tag{53b}$$

From these particular operators (53*b*) one recovers the whole quantiser by means of the covariance rule, using the usual matrix functions S which express the relativistic invariance of the Dirac equation; note that the formulae (53) are valid for any choice of the γ matrices. Note also that $\Omega_D(u)P_D = P_D\Omega_D(u)$.

The Wigner function corresponding to a Dirac wavefunction Ψ (in momentum space) is again simply the expected value of the quantiser Ω_D:

$$W_\Psi(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) = (\Psi|\Omega_D(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})\Psi) = \int_{H_m^+} \bar{\Psi}(\xi)\gamma^0[\Omega_D(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n})\Psi](\xi)\, \mathrm{d}\mu(\xi).$$

For instance, if Ψ is the plane wave $\Psi_{k,1/2}$, we get

$$\begin{aligned}
&W_{k,1/2}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) \\
&\quad = \tfrac{1}{2}k^0\delta(\boldsymbol{p} - \boldsymbol{k})((10)(\bar{k}/m)^{1/2} \quad (10)(k/m)^{1/2})\begin{pmatrix} (k/m)^{1/2} & 0 \\ 0 & (\bar{k}/m)^{1/2} \end{pmatrix} \\
&\quad\quad \times \begin{pmatrix} \frac{1}{2}(1, \sqrt{3}\boldsymbol{n}) & 0 \\ 0 & \frac{1}{2}(1, \sqrt{3}\boldsymbol{n}) \end{pmatrix} \begin{pmatrix} (\bar{k}/m)^{1/2} & 0 \\ 0 & (k/m)^{1/2} \end{pmatrix} \begin{pmatrix} (k/m)^{1/2}\binom{1}{0} \\ (\bar{k}/m)^{1/2}\binom{1}{0} \end{pmatrix} \\
&\quad = \tfrac{1}{2}k^0\delta(\boldsymbol{p} - \boldsymbol{k})((10) \quad (10))\begin{pmatrix} \frac{1}{2}(1, \sqrt{3}\boldsymbol{n}) & 0 \\ 0 & \frac{1}{2}(1, \sqrt{3}\boldsymbol{n}) \end{pmatrix}\begin{pmatrix} \binom{1}{0} \\ \binom{1}{0} \end{pmatrix} \\
&\quad = \tfrac{1}{2}(1 + \sqrt{3}n_3)k^0\delta(\boldsymbol{p} - \boldsymbol{k}).
\end{aligned}$$

Analogously, $W_{k,-1/2}(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{n}) = \frac{1}{2}(1 - \sqrt{3}n_3)k^0\delta(\boldsymbol{p} - \boldsymbol{k})$.

In this paper we have treated only positive-energy particles. To repeat the foregoing development for negative-energy particles, which unfolds in a completely parallel way, one starts from the coadjoint orbit $\mathcal{O}_{m,s-}$ determined by replacing $h = (m^2 + \boldsymbol{p} \cdot \boldsymbol{p})^{1/2}$ by

$h = -(m^2 + p \cdot p)^{1/2}$. In particular, we remark that the negative-energy solutions of the Dirac equation arise in a similar manner. We obtain a Stratonovich–Weyl quantiser with values which are operators on the representation space $\mathscr{H}_m^{1/2,-}$, and an isomorphism of this space with a subspace $\hat{\mathscr{H}}_m^{1/2,-}$ of 4-spinors, corresponding to (50); the analogue of (53) gives rise to the desired quantiser for the space of negative-energy solutions. We leave the details to the reader.

As another useful exercise, we point out that, by putting back the c in our formulae, the whole formulation reduces to the Galilean one with $u = mc^2$ for the internal energy in the non-relativistic limit.

8. Concluding remarks

As a rule of thumb, contributions in the physical and mathematical literature have respectively tried to 'make relativistic' two different elements of phase-space quantum mechanics. On the physical side, it so happens that 'relativistic Wigner functions' have been sporadically employed for some time [33]; they are introduced by formally extending Wigner's definition [3] to Minkowskian phase spaces. On the mathematical side, the Weyl quantisation rule is perceived as the basic subject for generalisation and, besides the papers we comment on below, there has also been a busy Japanese school [34] trying to establish self-adjointness of some classes of operators obtained by formal application of the Weyl correspondence [2] in the relativistic context. Our carefully systematic generalisation allows at least a preliminary assessment of the worth of such attempts. In general, it would seem that the other elements of a proper Moyal formulation, such as the twisted product with its *tracial property*, cannot be appended to them; and no actual calculations in the Moyal spirit are done.

The basic definition for 'relativistic Wigner functions' from which most authors start is generally

$$N_\Phi(x, p) \propto \int \bar{\hat{\Phi}}(x+v)\hat{\Phi}(x-v) \exp[2i(pv)]\, d^4v$$

where $\hat{\Phi}$, the Fourier transform of Φ, is a wavefunction (or field) satisfying the Klein–Gordon equation. This is immediately seen to be a simple-minded generalisation of (19). Some fail to note that such an object N_Φ must then satisfy the equation

$$((pp) + m^2 + \Box_x) N_\Phi = 0$$

and so, unless Φ is a plane wave, in which case N_Φ would equal our W_Φ, N_Φ is not supported on the mass shell. In practice, this means that this 'transport approach' to relativistic field theory is of an approximate nature from the beginning, which we consider unwarranted.

In the more mathematical vein, there have been some recent attempts to generalise 'Weyl correspondences' to the relativistic context (always in the spinless case). Ali and Antoine [35] purport to have a recipe for a relativistic Weyl transform for the $(1+1)$ Poincaré group. Their approach, for all its mathematical sophistication, stems from the 'old' form of the Weyl rule, which is notoriously difficult to generalise, instead of the newer one afforded by the Grossmann–Royer operators. They obtain results different from ours. The Unterbergers [36] come nearer to our point of view, as they apply covariance and the heuristic parity rule, which is similar to ours but without the factor $\{p\xi\}^{3/2}$. Of course, the resulting correspondence rule has no tracial property; this is the reason why they have to define two symbols, a 'passive' and an 'active' one.

We want to emphasise that the bridge between the coadjoint orbits and the representation spaces, given by our Stratonovich–Weyl quantiser, must be carefully constructed in order to ensure the *physical equivalence* with the standard quantum theory. In this respect we depart from the *démarche* of the school-creating papers by Bayen *et al* [37] that give rise to a bewildering variety of 'twisted products' of little use in physics. What all the reviewed attempts have in common is a methodology in which an element of the ordinary Moyal formulation is detached from the rest and imputed an unrestricted power and significance in the generalisation; we have shown, nevertheless, that a procedure is available which combines harmoniously *all* the basic elements of the Moyal approach.

Besides [10, 11], there is not much precedent, then, in the literature for our endeavour. We want to point out, however, the formal analogy between our work and the 'discrete quantum mechanics' formalism in [38]. Also, the idea to start quantisation from elementary classical systems in our sense is present in the interesting paper [39].

Summarising, we have provided the foundations for the phase-space formulation of relativistic quantum theory. There is much work to do before the present approach can show its usefulness as an established method in elementary particle physics. To begin with, some obviously unfinished business remains, such as quantisation of observables in the Dirac case, constructions of SW quantisers for higher-spin wave equations, treatment of the orbits corresponding to massless particles, and so on. To include treatments of all these topics would have excessively lengthened a paper already not very brief.

The old discussion about localisation and position observables, which has sometimes been conducted at an appalling level, can very well be clarified by employing our one-to-one quantisation–dequantisation rule: proposed position operators can always be dequantised for examination of their reasonableness at the classical level, and vice versa. We plan to develop in a forthcoming paper how interactions can be introduced and the application of our theory in simplifying perturbative QED. Finally, our proof of existence and uniqueness of the quantiser is specific to the Poincaré group. An interesting mathematical question concerns the realisation of the Stratonovich–Weyl postulates for general classes of groups.

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